

C^1 -generic billiard tables have a dense set of periodic points

Marie-Claude ARNAUD ^{*†}

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Abstract

We prove that the set of periodic points of a generic C^1 -billiard table is dense in the phase space.

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*Avignon University, LMA, EA 2151 F-84000, Avignon, France .

†Membre de l'Institut universitaire de France

1 Introduction

In the general domain of conservative dynamical systems, periodic orbits have an important place. Some questions concerning them are :

1. Do they exist ?
2. What is their type (are they elliptic, hyperbolic...)?
3. How many are they (for example, what is the number of periodic orbits having less than N bounces?) ?
4. Is the set of periodic points dense in the considered manifold ?...etc ...

In the particular domain of smooth strictly convex billiard tables, some answers concerning the question 1) (the proof is due to Birkhoff ; see for example [8]), the question 2) (see [8]) and the question 3) (see [9]) exist. Some answer to 1) in the case of smooth non convex billiard is given in [3]. But nowhere the case of the density of periodic points in smooth tables is treated. To be complete, let us mention an example of a C^∞ billiard table in [12] where the dynamics is formally conjugated to an irrational rigid rotation on some open set: if the conjugacy could be a true conjugacy, the billiard map has an open set without periodic point.

However, in the domain of conservative dynamical systems (symplectic diffeomorphisms for example), it has been proved in [10] (see [1] too for a slightly different proof) that every C^1 -generic symplectic diffeomorphism of a compact manifold has a dense subset of periodic points. This result is the consequence of a hard theorem, called the “ C^1 -closing lemma”. The C^0 -version of this result is quite easier. Let us explain in few words the proof of the “ C^0 -closing lemma ” :

C^0 -closing lemma : *Let $f : M \rightarrow M$ be a homeomorphism of a manifold M and $x \in M$ a positively recurrent point for f . Let \mathcal{U} be a C^0 - neighbourhood of f . Then there exists $g \in \mathcal{U}$ such that x is periodic for g .*

Proof : There exists a connected open neighbourhood V of x such that :

“if $g : M \rightarrow M$ is a homeomorphism and if $\text{support}(g \circ f^{-1}) \subset V$ then $g \in \mathcal{U}$ ”

The point x being recurrent, there exists $N \geq 1$ such that $f^N x \in V$. We choose N as small as possible. There are two cases :

- x is periodic for f ; we choose $g = f$.
- x is not periodic for f . Then V is a connected neighbourhood of x and $f^N x$. There exists $h : M \rightarrow M$ homeomorphism such that $h(f^N x) = x$ and $\text{support}(h) \subset V$.

If we define g by $g = h \circ f$, then :

- $\text{support}(g \circ f^{-1}) = \text{support}(h) \subset V$, thus $g \in \mathcal{U}$;
- $f(x) \notin \text{support}(h), \dots, f^{N-1}(x) \notin \text{support}(h)$. Thus $g(x) = h \circ f(x) = f(x)$,
... , $g^{N-1}(x) = g \circ g^{N-2}(x) = g(f^{N-2}(x)) = h(f^{N-1}(x)) = f^{N-1}(x)$ and :
 $g^N(x) = h \circ f^N(x) = x$. Therefore x is N -periodic for g .

□

The previous proof uses a fundamental argument : “if V is a small enough connected open set, if $x, y \in V$, there exists h homeomorphism such that $\text{support}(h) \subset V$ and

$h(x) = y$ ". Let us explain why this kind of argument doesn't work for billiards : assume you change a small part of the billiard table near a point x_0 , then you have changed the billiard map in a "large" open set, the set U of all rays coming from any bounce point that is close to x_0 : the bounce point x is close to x_0 , but the direction of the ray may be arbitrarily chosen. The problem is that we have a fibered system : if you change something, you change it along a whole fiber. This problem occurs for all fibered problems: geodesic flows, mechanical systems... and that is why we need different arguments for these cases. The problem of closing one orbit for a geodesic flow was recently solved by L. Rifford in [11].

Without asking exactly a closing lemma, we may ask ourselves :

"is the set of periodic points dense for a "general" smooth billiard table ? "

We obtain a positive answer in the category of C^1 -billiard tables :

Theorem 1 *There exists a dense G_δ subset (for the C^1 topology) of the set of C^1 -billiard tables such that, for every billiard table of this G_δ subset, the set of periodic points for the billiard map is dense in the phase space.*

Let us notice that we are unable to prove a similar result in the category of convex C^1 -billiard tables; this is a little surprising, because a classical argument due to Birkhoff prove that every convex billiard table has an infinity of periodic orbits and this result is not known for non-convex billiard tables!

We will precisely define the considered sets and topologies in the section 2. Let us just remark that if we perturb the billiard table in C^1 topology, we perturb the billiard map in C^0 topology.

Let us explain what is the main ingredient of the proof : in [4], the authors prove that every rational polygonal billiard has a dense set of periodic points. Thus the main idea will be : approximating a piecewise C^1 curve by a rational polygonal one, we will create some periodic points. Using another small perturbation, we smooth the rational polygon at some corners and make the new periodic points non degenerate (that means that if we do another small perturbation, these new periodic points will still exist). Then a classical Baire argument is sufficient to obtain the conclusion.

2 Definitions and results concerning the topology

We define $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$ and Γ the set of C^1 -maps $\gamma : \mathbf{S}^1 \rightarrow \mathbf{R}^2$ endowed with the C^1 -norm (here $\|\cdot\|_\infty$ is the norm "sup" associated to the usual Euclidian norm of \mathbf{R}^2) :

$$\|\gamma\| = \|\gamma\|_\infty + \|\gamma'\|_\infty .$$

It is well-known that $(\Gamma, \|\cdot\|)$ is a Banach space. We denote by d the associated distance. We define then the set \mathcal{N} of normal parametrisations of loops with length 1 :

$$\mathcal{N} = \{\gamma \in \Gamma ; \forall t \in \mathbf{S}^1, \|\gamma'(t)\| = 1\} .$$

Then \mathcal{N} is a closed subset of Γ , and (\mathcal{N}, d) is complete. Let us define on \mathcal{N} an equivalence relation \sim by : $\gamma_1 \sim \gamma_2$ if there exists an isometry $u : \mathbf{R} \rightarrow \mathbf{R}$ such that $\gamma_1 = \gamma_2 \circ u$. If $\mathcal{L} = \mathcal{N} / \sim$, we define on \mathcal{L} :

$$\delta(\ell_1, \ell_2) = \inf\{d(\gamma_1, \gamma_2); \gamma_1 \in \ell_1, \gamma_2 \in \ell_2\} .$$

It is easy to see that δ is a distance on \mathcal{L} and that (\mathcal{L}, δ) is complete.

The subset \mathcal{U} of \mathcal{L} defined by :

$$\mathcal{U} = \{\ell \in \mathcal{L} ; \exists \gamma \in \ell, \quad \forall t_1, t_2 \in \mathbf{S}^1, t_1 \neq t_2 \implies \gamma(t_1) \neq \gamma(t_2)\}$$

is an open subset of \mathcal{L} and therefore (\mathcal{U}, d) is a Baire space. We can identify it with the set of 1-dimensional compact C^1 submanifolds of \mathbf{R}^2 with length 1.

Now we can define a billiard table ; a *billiard table* B is the closure of the bounded connected component of the image of an element ℓ of \mathcal{U} ; then we have $\partial B = \text{Im } \ell$ and B is a simply connected 2-dimensional manifold with boundary. We name \mathcal{B} the set of billiard tables, that is too the set of simply connected 2-dimensional compact C^1 -submanifold with boundary of \mathbf{R}^2 whose boundary has its length equal to 1. As the map $\Phi : \ell \in \mathcal{U} \longrightarrow B \in \mathcal{B}$ is a bijection, we can define a metric Δ on \mathcal{B} by :

$$\forall (B_1, B_2) \in \mathcal{B}^2, \quad \Delta(B_1, B_2) = \delta(\phi^{-1}(B_1), \phi^{-1}(B_2))$$

and we know that (\mathcal{B}, Δ) is a Baire space ; more precisely, (\mathcal{B}, Δ) is an open subset of a complete metric space, and then there exists a metric Δ' on \mathcal{B} which is topologically equivalent to Δ such that (\mathcal{B}, Δ') is complete.

Let us now give a lemma that is interesting to understand the topology of \mathcal{B} and that we will use later in the proof of theorem 1.

Lemma 2 *Let d_H be the Hausdorff distance defined on the set \mathcal{K} of non-empty compact subsets of $T\mathbf{R}^2 = \mathbf{R}^2 \times \mathbf{R}^2$. For every $B \in \mathcal{B}$, let us denote the unitary tangent fiber bundle of ∂B by $K(B)$. Then if we define on \mathcal{B} the metric α by :*

$$\alpha(B_1, B_2) = d_H(K(B_1), K(B_2))$$

then α is topologically equivalent to Δ .

REMARK : Thus the considered topology is just the one associated to the Hausdorff metric in the tangent fiber bundle. Let us notice that the result is true even if we replace the unitary tangent fiber bundle of ∂B by the unitary tangent fiber bundle of B (which is a 3-dimensional submanifold with boundary of $T\mathbf{R}^2$)

In a Baire space, any countable intersection of open dense subsets is dense. We call *generic* a property which is verified by all the elements of such a set (i.e. a countable intersection of open dense subsets). Then any countable intersection of generic properties is a generic property and a generic property of a Baire space is satisfied by a dense subset of the Baire space. We will work in the Baire space (\mathcal{B}, α) .

Some other spaces of billiards are interesting too. The first one, named \mathcal{P} , is the set of piecewise C^1 -billiard tables with length 1. Then \mathcal{B} is a subset of \mathcal{P} , but there exist many elements in $\mathcal{P} \setminus \mathcal{B}$, as the polygonal simply connected billiards with length 1. It is easy to define a metric α on \mathcal{P} whose restriction to \mathcal{B} is the distance α that was defined in lemma 2. If \mathcal{R} is the subset of \mathcal{P} whose elements are the rational simply connected polygons with length 1 (a polygon is rational if all its angles are rational multipliers of π), it is easy too to see that :

Lemma 3 \mathcal{R} is dense in \mathcal{P} . Therefore, \mathcal{B} is contained in the closure of \mathcal{R} .

We will use the previous lemma to approximate smooth billiards by rational polygonal ones, and to apply some results concerning polygonal rational billiards.

REMARK :1. \mathcal{R} and \mathcal{P} are not Baire spaces. In these spaces, we couldn't do similar proofs. For example, we are unable to decide if a "generic" polygonal billiard has a periodic orbit.
 2. We will denote by \mathcal{P}_N the set of billiards tables that have at most N corners.

3 Definitions and results concerning billiard maps

Now, we consider $B \in \mathcal{P}$. Let $F = \{x_1, \dots, x_N\} \subset \mathbf{R}^2$ be the (finite) set of corners of ∂B . At every $x \in \partial B \setminus F$, we can define the tangent space $D(x)$ to ∂B at x . Then we define $F(x)$ as being the set of unitary vectors $v \in \mathbf{R}^2 = T_x \mathbf{R}^2$ which are on the other side of $D(x)$ than B : in fact, $F(x)$ is a closed half-circle. Then $\Sigma = \Sigma(B) = \bigcup_{x \in \partial B \setminus F} F(x)$ is a

2-dimensional topological manifold with boundary.

If $x \in \partial B \setminus F$ and $v \in D(x) \cap F(x)$, we define : $b(x, v) = (x, v)$. If $x \in \partial B \setminus F$ and $v \in F(x) \setminus D(x)$, we define :

- w is the image of v by the reflection of line $D(x)$;
- $y = x + \lambda w$ where $\lambda = \inf\{t > 0; x + tw \in \partial B\}$;
- $b(x, v) = (y, w)$.

$b = b_B$ is called the *billiard map*.

REMARK :Our definition of billiard map is not exactly the usual one; more precisely, there exist two involutions I_1 and I_2 such that $b = I_1 \circ I_2$ and what is usually called the billiard map is in fact $I_2 \circ I_1 = I_1 \circ b \circ I_1^{-1}$; therefore the two maps are conjugated one to each other and they have the same dynamical behavior. Moreover, our definition has the following advantage : b is defined exactly on the set Σ .

When B is not convex, b is not continuous. But it is continuous at every point (x, v) such that $p_2 \circ b(x, v)$ is not tangent to ∂B at $p_1 \circ b(x, v)$ (where we define : $p_1(x, v) = x$ and $p_2(x, v) = v$).

The following result is proved in [4] :

Theorem 4 (*Boshernitzan, Galperin, Kruger, Troubetzkoy*) : Let $B \in \mathcal{R}$ be a rational polygonal billiard; then the set of periodic points of b_B is dense in $\Sigma(B)$.

We will use this result in the next section to prove theorem 1.

4 Proofs of theorem 1

Let $(U_n)_{n \in \mathbf{N}}$ be a countable basis of open subsets of $T^1 \mathbf{R}^2$, the unitary tangent fiber bundle of \mathbf{R}^2 . Then we define a family $(Q_n)_{n \in \mathbf{N}}$ of properties on \mathcal{P} by :

" $P \in \mathcal{P}$ satisfies Q_n if one of the following situations happens :

1. $\Sigma(P) \cap U_n = \emptyset$;

2. $\Sigma(P) \cap U_n$ contains a periodic point for $b = b_P$ ".

If we prove that the set $\mathcal{Q}_n = \{P \in \mathcal{B}; P \text{ verifies } Q_n\}$ contains an open dense subset of \mathcal{B} , then theorem 1 is proved.

Thus let $n \in \mathbf{N}$ be fixed and $\mathcal{U} \subset \mathcal{B}$ be a non-empty open subset of \mathcal{B} . We have to prove that the interior of $\mathcal{U} \cap \mathcal{Q}_n$ is non-empty. Two cases are possible :

- either $\forall P \in \mathcal{U}, \Sigma(P) \cap U_n = \emptyset$; then $\mathcal{U} \subset \mathcal{Q}_n$ and $\mathcal{U} \cap \mathcal{Q}_n = \mathcal{U}$ has a non-empty interior;
- or there exists $B_0 \in \mathcal{U}$ such that $\Sigma(B_0) \cap U_n \neq \emptyset$. Then, $\mathcal{U}' = \{B \in \mathcal{U}; \Sigma(B) \cap U_n \neq \emptyset\}$ is a non-empty open subset of \mathcal{U} . There exists \mathcal{V} open subset of \mathcal{P} such that $\mathcal{V} \cap \mathcal{B} = \mathcal{U}'$. Moreover, we can ask that : $\forall P \in \mathcal{V}, P \cap U_n \neq \emptyset$ (because the condition " $P \cap U_n \neq \emptyset$ " is open) and that $\mathcal{V} \supset \{B \in \mathcal{P}; \alpha(K(B_0), K(B)) < \delta\}$ for some $\delta > 0$ where the distance α was defined in lemma 2. We have seen in section 2 that \mathcal{R} is dense in \mathcal{P} . Then there exists $P_0 \in \mathcal{R}$ such that $\alpha(K(B_0), K(P_0)) < \frac{\delta}{10}$. Because $\alpha(K(B_0), K(P_0)) < \frac{\delta}{10}$, P_0 has m corners z_1, \dots, z_m and at these corners the distance of the two unitary tangent vectors is less than $\frac{\delta}{5}$. As P_0 is a rational polygonal billiard, we can use theorem 4 : the billiard map b_0 associated to the billiard table P_0 has a periodic point $(x_0, v_0) \in U_n$.

Because (x_0, v_0) is a periodic point of b_0 , its (finite) orbit under b_0 doesn't contain any vertex. We can smooth the billiard table near the vertices z_1, \dots, z_p without losing the fact that (x_0, v_0) is periodic; using a small homothety (to be sure to obtain a length equal to 1), we obtain a C^2 -billiard table P_1 such that $(x_0, v_0) \in U_n \cap \partial P_1$ is periodic for the billiard map b_1 associated to P_1 . Because $\alpha(K(B_0), K(P_0)) < \frac{\delta}{10}$ and because the distance of the two unitary tangent vectors at these corners of P_0 is less than $\frac{\delta}{5}$, we can ask that $\alpha(K(B_0), K(P_1)) < \delta$ and then that $P_1 \in \mathcal{V} \cap \mathcal{B} = \mathcal{U}'$.

We have not finished the proof of theorem 1 because we have find $P_1 \in \mathcal{U} \cap \mathcal{Q}_n$ but we don't know if $\mathcal{U} \cap \mathcal{Q}_n$ has a non-empty interior. The next idea is then to perturb P_1 in such a way that (x_0, v_0) becomes stably periodic.

To do that, we recall some results contained in [8]; we will call $(x_1, v_1) = b_1(x_0, v_0), \dots, (x_\tau, b_\tau) = b_1^\tau(x_0, v_0) = (x_0, v_0)$ the points of the (periodic) orbit of (x_0, v_0) under b_1 . Let us name (y_i, w_i) coordinates near (x_i, v_i) . In a neighbourhood of (x_i, v_i) , the map $((y_i, w_i) \rightarrow (y_i, z_i))$ where $z_i = p_1 \circ b_1(y_i, w_i)$ is a C^1 -diffeomorphism. Then, $(x_0, \dots, x_{\tau-1})$ (abbreviation for $((x_0, x_1), (x_1, x_2), \dots, (x_{\tau-1}, x_0))$ in these coordinates) is a periodic orbit if and only if it is a critical point of the C^2 length function defined by (we note $y_\tau = y_0$ in this case) :

$$\ell(y_0, \dots, y_{\tau-1}) = \sum_{i=1}^{\tau} \|y_i - y_{i-1}\|.$$

At such a point $(x_0, \dots, x_{\tau-1})$, we define (eventually in charts) :

$$a_i = \frac{\partial^2 \ell}{\partial y_i^2}(x_0, \dots, x_{\tau-1}) \text{ and } b_i = \frac{\partial^2 \ell}{\partial y_i \partial y_{i+1}}(x_0, \dots, x_{\tau-1}).$$

Then the Hessian of ℓ is :

- if $\tau = 2$: $H_2 = \begin{pmatrix} a_1 & b_1 + b_2 \\ b_2 + b_1 & a_2 \end{pmatrix}$;

• il $\tau > 2$: $H_\tau = \begin{pmatrix} a_1 & b_1 & 0 & \dots & b_\tau \\ b_1 & a_2 & b_2 & \dots & 0 \\ 0 & b_2 & a_3 & \dots & 0 \\ \dots & & \dots & & \dots \\ b_\tau & 0 & 0 & \dots & a_\tau \end{pmatrix}$.

An easy calculus made in [8] shows that :

- b_i depends only on x_i, x_{i+1} and the tangent space to ∂P_1 at x_i and x_{i+1} ;
- if the tangent space to ∂P_1 at x_i is fixed, a_i depends linearly on the curvature of ∂P_1 at x_i , and this dependance is effective (a_i is not constant).

As it is easy to change the curvature of a curve near a point without changing the tangent space at this point or far away this point, we can perturb P_1 in $P_2 \in \mathcal{B} \cap \mathcal{U}$ that is C^2 and such that : $\det H_\tau \neq 0$; indeed, $\det H_\tau$ is a non constant polynomial function in a_1, \dots, a_τ . For this new billiard P_2 , $(x_1, \dots, x_{\tau-1})$ is a non-degenerate critical point of the C^2 function ℓ_2 . We have :

Lemma 5 *There exists a neighbourhood W of ℓ_2 in the C^1 topology such that every element of W has a critical point near $(x_0, \dots, x_{\tau-1})$.*

REMARK :the previous lemma is more known in the case of the C^2 -topology. In the case of the C^2 -topology, we can even ask that the critical point near $(x_0, \dots, x_{\tau-1})$ is unique. In the C^1 -topology, we may have an infinity of such critical points, but we know that at least one of these critical points exists. Let us explain why.

If we consider ℓ that is C^2 but just C^1 close to ℓ_2 :

- we use the existence of an isolating block B for $\text{grad}\ell$, which is stable by C^1 -perturbation (see [5]); this implies the existence of one positive (or negative for a minimum) orbit for the flow (φ_t) of ℓ which stays in B ;
- but if ℓ has no critical point in B , this is impossible, because there exists a constant $k > 0$ such that : $\forall x \in B, \frac{d\ell \circ \varphi_t}{dt}(x) = \|\text{grad}\ell(x)\|^2 \geq k$, and then $\lim_{t \rightarrow +\infty} \ell \circ \varphi_t(x) = +\infty$ (and $\lim_{t \rightarrow -\infty} \ell \circ \varphi_t(x) = -\infty$) and thus the orbit leaves B .

If ℓ is not C^2 , we cannot use the same argument because the gradient flow is not defined. Let us assume that ℓ is C^1 close to ℓ_2 and has no critical point in the isolating block. By using a convolution, we can approximate ℓ in C^1 topology by a smooth ℓ_1 that has no critical point in the isolating block, and this contradicts what we explained for C^2 functions ℓ .

Now, we can finish the proof of theorem 1 : every billiard table B C^1 -close to P_2 has its length function ℓ which is C^1 -close to ℓ_2 ; thus the associated billiard map has at least one periodic orbit close to $((x_0, v_0), \dots, (x_{\tau-1}, v_{\tau-1}))$ and therefore a periodic point in U_n . Then a whole neighbourhood of P_2 in \mathcal{B} is in \mathcal{Q}_n and then the interior of $\mathcal{U} \cap \mathcal{Q}_n$ is non-empty. \square

REMARK :1. The problem of convex billiards is less easy. Of course, we can prove that their set is a Baire set and we can approximate any convex C^1 -billiard table by a convex rational polygonal one having a periodic orbit; we can smooth this billiard to obtain a convex one; but if the periodic orbit of the rational polygonal billiard has more than one bounce point on every side, you cannot perturb it in such a way that you obtain a strictly convex C^2 -billiard table having the same periodic orbit. And if the smooth, the billiard

table that we obtain is convex but not strictly convex, we cannot ask, when we perturb it in the C^2 -topology to obtain a stable periodic orbit, that the new billiard table is convex.

2. We know that a generic element of \mathcal{B} is topologically transitive (see [6] for a proof in a slightly different topology, but easily transposable in our topology by using [7]). A dynamical system having a dense subset of periodic points and being topologically transitive is called chaotic (see [2]); we have then prove that the billiard map associated to a generic C^1 billiard table is chaotic.

References

Bibliography.

1. M.-C. Arnaud. Le “Closing Lemma” en topologie C^1 , *Mem. Soc. Math. Fr.* **74** (1998).
2. J. Banks; J. Brooks; G. Cairns; G. Davis; P. Stacey. *On Devaney’s definition of chaos*. *Amer. Math. Monthly* **99** (1992), no. 4, 332–334.
3. V. Benci; F. Giannoni. *Periodic bounce trajectories with a low number of bounce points*. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **6** (1989), no. 1, 73–93.
4. M. Boshernitzan; G. Galperin; T. Kruger; S. Troubetzkoy. *Periodic billiard orbits are dense in rational polygons*. *Trans. Amer. Math. Soc.* **350** (1998), no. 9, 3523–3535.
5. C. Conley. Isolated invariant sets and the Morse index, *C.B.M.S, Reg. Conf. series in Math.* **38** (1978).
6. P. M. Gruber. *Convex billiards*. *Geom. Dedicata* **33** (1990), no. 2, 205–226
7. A. B Katok & A. N Zemljakov. *Topological transitivity of billiards in polygons*. (Russian) *Mat. Zametki* **18** (1975), no. 2, 291300.
8. V. Kozlov & D. Treshchev. *Billiards : a genetic introduction in the dynamics of systems with impacts*, *Transl. of math. Monograph. of A.M.S.* **89** (1991).
9. H. Masur. *The growth rate of trajectories of a quadratic differential*. *Ergodic Theory Dynam. Systems* **10** (1990), no. 1, 151–176.
10. C. Pugh & C. Robinson. *The C^1 closing lemma, including Hamiltonians*, *Ergod. Th. & Dynam. Sys.* **3** (1983), 261-314.
11. L. Rifford, *Closing geodesics in C^1 topology*, *J. Differential Geom.*, **91(3)** (2012), 361-382.
12. D.Treschev, *Billiard map and rigid rotation*. *Physica D*, Volume **255**, (15 July 2013), 3134