

# A multidimensional birkhoff theorem for time-dependent tonelli hamiltonians

Marie-Claude Arnaud, Andrea Venturelli

► **To cite this version:**

Marie-Claude Arnaud, Andrea Venturelli. A multidimensional birkhoff theorem for time-dependent tonelli hamiltonians: Birkhoff theorem. 2016. <hal-01309652v2>

**HAL Id: hal-01309652**

**<https://hal-univ-avignon.archives-ouvertes.fr/hal-01309652v2>**

Submitted on 16 Aug 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A multidimensional Birkhoff Theorem for time-dependent Tonelli Hamiltonians

Marie-Claude Arnaud<sup>†</sup> · Andrea Venturelli

Received : April 2016 / Revised version : March 2017

**Abstract** Let  $M$  be a closed and connected manifold,  $H : T^*M \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$  a Tonelli 1-periodic Hamiltonian and  $\mathcal{L} \subset T^*M$  a Lagrangian submanifold Hamiltonianly isotopic to the zero section. We prove that if  $\mathcal{L}$  is invariant by the time-one map of  $H$ , then  $\mathcal{L}$  is a graph over  $M$ .

An interesting consequence in the autonomous case is that in this case,  $\mathcal{L}$  is invariant by all the time  $t$  maps of the Hamiltonian flow of  $H$ .

**Keywords** Lagrangian Dynamics, Weak KAM Theory, Lagrangian submanifolds, generating functions.

**Mathematics Subject Classification (2010)** 37J50, 70H20, 53D12

## 1 Introduction and Main Results.

A lot of problems coming from the physics are conservative, as the  $N$ -body problem and other classical mechanical systems: in other words, they are symplectic.

Close to the completely elliptic periodic orbits of symplectic dynamics, it is in general possible to use some change of coordinates called normal form (see [19]) and thus to be led to study a local diffeomorphism

$$(\theta, r) \in \mathbb{T}^n \times \mathbb{R}^n \mapsto (\theta + \alpha + \beta.r, r) + \text{small} \quad (1)$$

The authors are supported by ANR-12-BLAN-WKBHJ and by MATH-AmSud project SIDIHAM

<sup>†</sup> member of the *Institut universitaire de France*.

M.-C. Arnaud  
Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151)  
F-84018 Avignon  
E-mail: marie-claude.arnaud@univ-avignon.fr

A. Venturelli  
Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151)  
F-84018 Avignon  
E-mail: andrea.venturelli@univ-avignon.fr

close to the zero section  $\mathbb{T}^n \times \{0\}$  with  $\beta$  being a symmetric matrix. When  $\beta$  is a definite matrix, these diffeomorphisms are called twist maps and it can also be proved that they are the time 1 map of a so-called Tonelli Hamiltonian vector field.

This kind of diffeomorphisms was introduced for example by Poincaré in the study of the circular restricted 3-body problem. When  $n = 1$ , they were intensively studied by G.D. Birkhoff. In [7], G.D. Birkhoff proved that if  $\gamma$  is an embedded circle of  $\mathbb{T} \times \mathbb{R}$  that is not homotopic to a point and that is invariant by some conservative twist map, then  $\gamma$  is the graph of a Lipschitz map  $\mathbb{T} \rightarrow \mathbb{R}$ . A modern proof of this result can be found in [14].

**Question:** what happens in higher dimensions?

A natural extension of the 1-dimensional annulus  $T^*\mathbb{T} = \mathbb{T} \times \mathbb{R}$  is the cotangent bundle  $T^*M$  of a closed  $n$ -dimensional manifold  $M$ . We recall in section 1.1 that  $T^*M$  can be endowed with a symplectic form.

If we want to obtain some submanifolds that are graphs (or more correctly sections) in  $T^*M$ , we are led to look at  $n$ -dimensional submanifolds. Moreover, we have to impose some topological conditions for these submanifolds. Indeed, there are examples of conservative twist maps of  $\mathbb{T} \times \mathbb{R}$  that have an invariant embedded circle that is homotopic to a point (and then this is not a graph): this happens for example for the time 1 map of the rigid pendulum close to the elliptic equilibrium.

But even if we ask that the invariant submanifold is homotopic to the zero-section of  $T^*M$ , it is easy to build examples of Tonelli Dynamics that have an invariant submanifold that is not a graph but is homotopic to the zero section. The first author gave in [2] an example of such a submanifold of  $T^*\mathbb{T}^3 = \mathbb{T}^3 \times \mathbb{R}^3$  that is invariant by the Hamiltonian flow of  $H(q, p) = \frac{1}{2}\|p\|^2 = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)$ , which is the geodesic flow for the flat metric on  $\mathbb{T}^3$ .

That is why we focus on the particular case of Lagrangian submanifolds.

**DEFINITION.** A submanifold  $\mathcal{L} \subset T^*M$  is *Lagrangian* if  $\dim \mathcal{L} = n$  and  $\omega|_{T\mathcal{L}} = 0$ .

Even if the set of Lagrangian submanifolds is very small in the set of all the  $n$ -dimensional submanifolds (more precisely it has no interior when endowed with the Hausdorff topology in  $T(T^*M)$ ), there exist a lot of invariant Lagrangian submanifolds for the symplectic dynamics.

**EXAMPLES.**

- In  $T^*\mathbb{T}$ , a loop is always Lagrangian;
- a vertical fiber  $T_q^*M$  is Lagrangian;
- the zero-section is Lagrangian;
- more generally, a  $C^1$  graph is Lagrangian iff it is the graph of a closed 1-form: for example,  $\{(q, dS(q)); q \in M\}$  is a Lagrangian submanifold;
- the stable or unstable (immersed) submanifold at a hyperbolic equilibrium is Lagrangian;
- for the so-called completely integrable systems the phase space is foliated by invariant Lagrangian tori;
- some of these invariant tori remain after perturbation (K.A.M. theory).

Let us come back to the expression (1). When  $\beta$  is indefinite, M. Herman constructed in [16] some examples with an invariant Lagrangian torus that is isotopic to the zero section but not a graph. That is why we will assume that  $\beta$  is positive definite, i.e. we will work with Tonelli Hamiltonians.

DEFINITION. A  $C^2$  function  $H : T^*M \times \mathbb{T} \rightarrow \mathbb{R}$  is a *Tonelli Hamiltonian* if

- the Hamiltonian vector field<sup>1</sup> associated to  $H$  is complete;
- $H$  is  $C^2$ -convex in the fiber direction, i.e. has a positive definite Hessian in the fiber direction;
- $H$  is superlinear in the fiber direction, i.e. for every  $B > 0$ , there exists  $A > 0$  such that:

$$\forall (q, p, t) \in T^*M \times \mathbb{T}, \|p\| \geq A \Rightarrow H(q, p, t) \geq B\|p\|.$$

EXAMPLE. A Riemannian metric defines an autonomous Tonelli Hamiltonian.

Let us recall that we need to ask some topological condition on the invariant Lagrangian submanifold to be able to prove that it is a graph. To explain that, we need a definition.

DEFINITION. Two submanifolds  $\mathcal{L}_1$  and  $\mathcal{L}_2$  of  $T^*M$  are *Hamiltonianly isotopic* if there exists a time-dependent Hamiltonian  $H : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$  such that, if  $(\phi_H^{s,t})$  is the family of symplectic maps that is generated by the Hamiltonian vector field of  $H$ , then

$$\mathcal{L}_2 = \phi_H^{0,1}(\mathcal{L}_1).$$

Our main result is the following one.

**Main Theorem** *Let  $M$  be a closed manifold, let  $H : T^*M \times \mathbb{T} \rightarrow \mathbb{R}$  be a Tonelli 1-time periodic Hamiltonian, and let  $\mathcal{L} \subset T^*M$  be a  $C^1$  Lagrangian submanifold Hamiltonianly isotopic to a Lagrangian graph. If  $\mathcal{L}$  is invariant by the time one map associated to  $H$ , then  $\mathcal{L}$  is the Lagrangian graph of a  $C^1$  closed 1-form.*

A submanifold  $\mathcal{L}$  is Hamiltonianly isotopic to the zero-section if and only if the two submanifolds are isotopic in some particular subset  $\mathcal{E}$  of the set of Lagrangian submanifolds of  $T^*M$ , the set of the so-called *exact Lagrangian submanifolds*. Hence some related questions remain open.

### Questions

- Is the same conclusion true if we replace “Hamiltonianly isotopic” by ” isotopic”?
- Is the same conclusion true if we replace “Hamiltonianly isotopic” by ” homotopic”?

Let us mention some related existing results.

- In [1], the first author proved that if a Lagrangian submanifold that is Hamiltonianly isotopic to a Lagrangian graph is invariant by a autonomous Tonelli Hamiltonian flow, then it has to be a graph. In next Corollary, we will explain how our result improves this statement.

---

<sup>1</sup> This will be defined in section 1.1.

- In [4], P. Bernard and J. dos Santos extended this result in the autonomous case to the case of Lipschitz Lagrangian submanifolds.
- In [5] and [6], for  $M = \mathbb{T}^n$ , with a weaker topological assumption (they assume that  $\mathcal{L}$  is homologous to the zero-section) but a strong hypothesis on the restricted Dynamics (it is assumed to be chain-recurrent), M. Bialy and L. Polterovich obtain the same result.
- In [15], M. Herman proved a similar result for a submanifold that is:
  - compact and Lagrangian;
  - with a Maslov class equal to 0;
  - invariant by an exact symplectic twist map of  $\mathbb{T}^n \times \mathbb{R}^n$  that is  $C^1$ -close enough to a completely integrable symplectic twist map;
  - such that the restricted dynamics is chain recurrent.

Our result is valid on all cotangent bundles and doesn't assume any dynamical behaviour of the restriction to the invariant submanifold.

However, it cannot be extended to any twist map because we don't know if a general twist map (in any dimension) is the time 1 map of a Tonelli Hamiltonian (see [13] for an interesting discussion on this subject).

Some arguments of our proof are common with the proof of the autonomous case in [1], but not all. Moreover, even if some of our techniques come from weak KAM theory, we tried to avoid to use the whole theory, as the Aubry set, the weak KAM solutions . . . and to write a self-contained article.

**Corollary 1** *Let  $M$  be a closed manifold, let  $H : T^*M \rightarrow \mathbb{R}$  be an autonomous Tonelli Hamiltonian, and let  $\mathcal{L} \subset T^*M$  be a  $C^1$  Lagrangian submanifold Hamiltonianly isotopic to a Lagrangian graph. If  $\mathcal{L}$  is invariant by the time one map associated to  $H$ , then  $\mathcal{L}$  is the Lagrangian graph of a  $C^1$  closed 1-form and it is invariant by all the time  $t$  maps  $\phi_H^t$ .*

To any Tonelli Hamiltonian  $H : T^*M \times \mathbb{T} \rightarrow \mathbb{R}$ , a Lagrangian function  $L : TM \times \mathbb{T} \rightarrow \mathbb{R}$  can be associated via the Legendre duality.

$$\forall (q, v, t) \in TM \times \mathbb{T}, L(q, v, t) = \inf_{p \in T_q^*M} (p \cdot v - H(q, p, t)).$$

DEFINITION. A continuous and piecewise  $C^1$  arc  $\gamma_0 : [a, b] \rightarrow M$  is *minimizing* if for every continuous and piecewise  $C^1$  arc  $\gamma : [a, b] \rightarrow M$  such that  $\gamma_0(a) = \gamma(a)$  and  $\gamma_0(b) = \gamma(b)$ , we have

$$\int_a^b L(\gamma_0(t), \dot{\gamma}_0(t), t) dt \leq \int_a^b L(\gamma(t), \dot{\gamma}(t), t) dt.$$

**Corollary 2** *Let  $M$  be a closed connected manifold,  $H : T^*M \times \mathbb{T} \rightarrow \mathbb{R}$  a Tonelli 1-time periodic Hamiltonian, and  $\mathcal{L} \subset T^*M$  a  $C^1$  Lagrangian submanifold Hamiltonianly isotopic to the zero section. If  $\mathcal{L}$  is invariant by the time one map associated to  $H$ , then the orbit of every point of  $\mathcal{L}$  is minimizing.*

## 1.1 Notations

- $M$  is a closed Riemannian manifold,  $\pi : T^*M \rightarrow M$  is its cotangent bundle and  $\mathcal{L}_{T^*M}$  the zero section; if  $q = (q_1, \dots, q_n)$  are coordinates in a chart of  $M$ , the dual coordinates  $p = (p_1, \dots, p_n) \in T_q^*M$  are defined by  $p_i(\delta q_j) = \delta_{i,j}$  where  $\delta q_i$  is the  $i$ th vector of the canonical basis and  $\delta_{i,j}$  is the Kronecker symbol;
- $T^*M$  is endowed with the Liouville 1-form that is defined by:

$$\forall p \in T^*M, \forall v \in T_p(T^*M), \lambda(v) = p \circ D\pi(p)(v);$$

in a dual chart, we have  $\lambda = \langle p, dq \rangle = \sum p_i dq_i$ ;

- the canonical symplectic form on  $T^*M$  is  $\omega = -d\lambda$ ; in a dual chart we have  $\omega = dq \wedge dp = \sum dq_i \wedge dp_i$ ;
- $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  is the 1-dimensional torus with length 1 and  $\mathbb{T}_2 = \mathbb{R}/2\mathbb{Z}$  is the 1-dimensional torus with length 2; we denote by  $t \in \mathbb{R} \mapsto [t]_1 \in \mathbb{T}$  and  $t \in \mathbb{R} \mapsto [t]_2 \in \mathbb{T}_2$  the corresponding covering maps;
- a complete  $C^2$  Hamiltonian  $H : T^*M \times \mathbb{T} \rightarrow \mathbb{R}$  being given, the Hamiltonian vector field  $X_H$  is defined by  $\omega(X_H(x, s), \delta x) = dH(x, s)\delta x$  and the corresponding Hamiltonian family of diffeomorphisms is denoted by  $(\phi_H^{s,t})_{s,t \in \mathbb{R}}$ .
- we choose coordinates  $(q, \tau)$  in the closed manifold  $\mathcal{M}_2 = M \times \mathbb{T}_2$  and denote the dual coordinates by  $(p, E)$ ; then the Liouville 1-form on  $T^*\mathcal{M}_2$  is  $\Theta = \langle p, dq \rangle + Ed\tau$  and the canonical symplectic form is  $\Omega = -d\Theta = dq \wedge dp + d\tau \wedge dE$ . We will often use the identification  $T^*\mathcal{M}_2 = T^*M \times \mathbb{T}_2 \times \mathbb{R}$ ;
- we choose similarly coordinates  $(q, \tau)$  in the closed manifold  $\mathcal{M}_1 = M \times \mathbb{T}$  and denote the dual coordinates by  $(p, E)$ ; then the Liouville 1-form on  $T^*\mathcal{M}_1$  is  $\theta = \langle p, dq \rangle + Ed\tau$ . We will often use the identification  $T^*\mathcal{M}_1 = T^*M \times \mathbb{T} \times \mathbb{R}$ .
- given a function  $v : \mathcal{M}_1 \rightarrow \mathbb{R}$  (resp.  $v : \mathcal{M}_2 \rightarrow \mathbb{R}$ ) and a point  $z = (q, t) \in \mathcal{M}_1$  (resp.  $z = (q, t) \in \mathcal{M}_2$ ), if  $v$  is differentiable on  $z$  we set

$$J_v(z) = (q, d_q v(z), t, \frac{\partial v}{\partial t}(z)).$$

It is an element of  $T^*M \times \mathbb{T} \times \mathbb{R}$  (resp.  $T^*M \times \mathbb{T}_2 \times \mathbb{R}$ ) and it can be identified with the differential of  $v$  at  $z = (q, t)$ .

## 1.2 A useful reduction

Let us explain why we will assume that  $\mathcal{L}$  is Hamiltonianly isotopic to the zero section (instead of “to a Lagrangian graph”) in the proof.

Assume that  $\mathcal{L}$  is Hamiltonianly isotopic to the Lagrangian graph  $\mathcal{L}_0$  and that  $\mathcal{L}$  is invariant by the time 1 map of the Tonelli Hamiltonian  $H : T^*M \times \mathbb{T} \rightarrow \mathbb{R}$ .

Then  $\mathcal{L}_0$  is the graph of some closed 1-form  $\Lambda$  of  $M$ . Changing  $\mathcal{L}_0$  in a very close other graph, we can even assume that  $\Lambda$  is smooth. Then  $F : T^*M \rightarrow T^*M$  that is defined by  $F(q, p) = (q, p + \Lambda(q))$  is a symplectic diffeomorphism so that

- $F^{-1}(\mathcal{L})$  is Hamiltonianly isotopic to the zero section  $F^{-1}(\mathcal{L}_0) = \mathcal{L}_{T^*M}$ ;
- $F^{-1}(\mathcal{L})$  is invariant by the time 1 map of the Tonelli Hamiltonian  $\tilde{H}(q, p, t) = H(F(q, p), t)$ .

Hence if we have proved the main theorem for the submanifolds that are Hamiltonianly isotopic to the zero section, we deduce that  $F^{-1}(\mathcal{L})$  and then  $\mathcal{L}$  is a graph.

### 1.3 Structure of the article

- In the second section, we construct an extended autonomous Hamiltonian and an extended Lagrangian submanifold in the extended phase space; we then build a graph selector for the extended Lagrangian submanifold;
- in section 3, using the graph selector that was built in section 2, we build a dominated function;
- in section 4, using the notion of calibrated curve, we prove the main theorem and its corollaries.

## 2 Construction of a Lagrangian submanifold in the extended phase space and its graph selector.

**Standing hypotheses** From now, we assume that  $H : T^*M \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$  is a Tonelli time 1-periodic Hamiltonian and that  $\mathcal{L} \subset T^*M$  is a  $C^1$  Lagrangian submanifold Hamiltonianly isotopic to  $\mathcal{L}_{T^*M}$  that is invariant by the time one map associated to  $H$ .

The goal of this section is to build an extended autonomous Hamiltonian, an extended Lagrangian submanifold and a so-called graph selector. Moreover, we will prove some properties for these objects.

### 2.1 Extension of the Lagrangian submanifold

In this section, adding two dimensions to the phase space, we will replace the non-autonomous Hamiltonian flow by an autonomous one and extend the invariant submanifold in the new phase space.

Let us comment on the choice of the new Hamiltonian and of the Lagrangian submanifold. The method that gives an autonomous Hamiltonian is well-known but the extended Hamiltonian is not Tonelli with respect to the new variables and we cannot just apply the proof that the first author gave in [1] in the autonomous case. The method to build an extended submanifold in the new phase-space is well-known too, but:

- a priori, this new submanifold has a boundary; thus we would need to build a theory of generating functions for manifolds with boundary to go on with our proof and we prefer to avoid this. Moreover, we don't know if this could work;
- an idea to remove the problem of boundary is to identify what happens for the times  $t = 0$  and  $t = 1$ . As the initial manifold is invariant, we can glue the two ends of the extended submanifold in a smooth way and obtain a closed manifold. Then a new problem appears: we cannot extend the isotopy that joins the zero

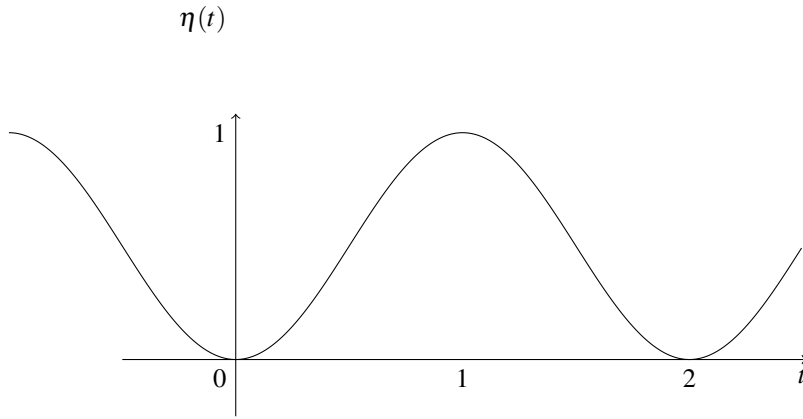
section to the initial manifold in a periodic way because the submanifolds that appear in the isotopy are not invariant by the initial time 1 map and thus their extended submanifolds cannot be glued in a continuous way.

To overcome these problems, we had the idea to extend the Hamiltonian flow in a 2-periodic one by symmetrizing the extended Hamiltonian and the extended submanifold. Let us explain this now.

We use the following function.

NOTATION. Let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative and  $C^\infty$  fonction satisfying the following properties

- i)  $\eta(-t) = \eta(t)$  and  $\eta(1-t) = \eta(1+t)$  for every  $t \in \mathbb{R}$ .
- ii)  $\eta(0) = 0$  and  $\eta(1) = 1$ .
- iii)  $\dot{\eta}(t) > 0$  if  $t \in (0, 1)$ .
- iv)  $\dot{\eta}(0) = \ddot{\eta}(0) = \dot{\eta}(1) = \ddot{\eta}(1) = 0$ .



Let us introduce a new time-dependent Hamiltonian  $K : T^*M \times \mathbb{T}_2 \rightarrow \mathbb{R}$  defined by  $K(q, p, t) = \dot{\eta}(t)H(q, p, \eta(t))$ . A straightforward computation shows that every integral curve of  $X_K$  can be written as  $t \mapsto (q, p)(t) = (Q, P)(\eta(t))$ , where  $s \mapsto (Q, P)(s)$  is an integral curve of  $X_H$ . This fact can be expressed as follows :

$$\varphi_K^{s,t}(q, p) = \varphi_H^{\eta(s), \eta(t)}(q, p).$$

Roughly speaking, integral curves of  $X_K$  are reparametrizations of segments of integral curves of  $X_H$ , but they slow down and turn back at integer time. In particular, integral curves of  $X_K$  are all 2-periodic and satisfy  $(q, p)(-t) = (q, p)(t)$  and  $(q, p)(1-t) = (q, p)(1+t)$ .

Let now  $\mathcal{H} : T^*M \times \mathbb{T}_2 \times \mathbb{R} \rightarrow \mathbb{R}$  be the autonomous Hamiltonian on the extended phase space  $T^*\mathcal{M}_2 = T^*M \times \mathbb{T}_2 \times \mathbb{R}$  defined by

$$\mathcal{H}(q, p, \tau, E) = K(q, p, \tau) + E.$$



The Hamiltonian equations for  $\mathcal{H}$  are given by

$$\begin{cases} \frac{dq}{dt} = \frac{\partial K}{\partial p}(q, p, \tau), & \frac{d\tau}{dt} = 1, \\ \frac{dp}{dt} = -\frac{\partial K}{\partial q}(q, p, \tau), & \frac{dE}{dt} = -\frac{\partial K}{\partial \tau}(q, p, \tau), \end{cases} \quad (2)$$

We can see that the evolution on  $T^*M$  is the same as for the Hamiltonian  $K$ , while the variable  $\tau$  is essentially the time. If  $\phi_{\mathcal{H}}^t$  denotes the Hamiltonian flow of  $\mathcal{H}$ , by (2), we easily get the relation

$$\phi_{\mathcal{H}}^t(q, p, \tau, E) = (\phi_K^{\tau, \tau+t}(q, p), \tau + t, E + K(q, p, \tau) - K(\phi_K^{\tau, \tau+t}(q, p), \tau + t)). \quad (3)$$

The evolution of the variable  $E$  can be obtained using the conservation of  $\mathcal{H}$  by the flow. Let  $\mathcal{G} \subset T^*M \times \mathbb{T}_2 \times \mathbb{R}$  be the submanifold defined by

$$\mathcal{G} = \{\phi_{\mathcal{H}}^t(q, p, 0, -K(q, p, 0)), (q, p) \in \mathcal{L}, t \in [0, 2]\}.$$

Since the flow  $\phi_{\mathcal{H}}^t$  is 2-periodic in time,  $\mathcal{G}$  is a closed submanifold diffeomorphic to  $\mathcal{L} \times \mathbb{T}_2$ . If we cut  $\mathcal{G}$  by a hypersurface  $\tau = t$  and we forget the  $E$  variable, we get the image of  $\mathcal{L}$  by  $\varphi_K^{0,t} = \varphi_H^{0,\eta(t)}$ .

**Proposition 1** *If  $\mathcal{L}$  is Hamiltonianly isotopic to the zero section  $\mathcal{Z}_{T^*M}$  of  $T^*M$ , then  $\mathcal{G}$  is Hamiltonianly isotopic to the zero section  $\mathcal{Z}_{T^*\mathcal{M}_2}$  of  $T^*\mathcal{M}_2$ .*

*Proof* The proof is twofold. In the first part, we prove that  $\mathcal{G}$  is isotopic to some submanifold  $\mathcal{G}_0$  by using the fact that  $\mathcal{L}$  and  $\mathcal{Z}_{T^*M}$  are isotopic. In the second part, we prove that  $\mathcal{G}_0$  is isotopic to  $\mathcal{Z}_{T^*\mathcal{M}_2}$  by using the time-dependent Hamiltonian  $s \mapsto sK$ .

Let  $(\psi_s)$  be a Hamiltonian isotopy of  $T^*M$  such that  $\psi_0 = \text{Id}_{T^*M}$  and  $\psi_1(\mathcal{Z}_{T^*M}) = \mathcal{L}$ . We use the notation  $\mathcal{L}_s = \psi_s(\mathcal{Z}_{T^*M})$ . We denote by  $h(q, p, s)$  a Hamiltonian associated to  $(\psi_s)$ .

For every  $s \in [0, 1]$ , we define the submanifold  $\mathcal{G}_s$  of  $T^*\mathcal{M}_2 = T^*M \times \mathbb{T}_2 \times \mathbb{R}$  by

$$\mathcal{G}_s = \{\phi_{\mathcal{H}}^t(q, p, 0, -K(q, p, 0)), (q, p) \in \mathcal{L}_s, t \in [0, 2]\}.$$

Exactly for the same reason as  $\mathcal{G}$ ,  $\mathcal{G}_s$  is a closed manifold that is diffeomorphic to  $\mathcal{L}_s \times \mathbb{T}_2$ .

Because  $t \mapsto \varphi_K^{0,t}$  is 2-periodic, we can use this notation for  $t \in \mathbb{T}_2$  too.

We define  $F_s : T^*\mathcal{M}_2 \rightarrow T^*\mathcal{M}_2$  by

$$F_s(q, p, \tau, E) = (\varphi_K^{0,\tau} \circ \psi_s \circ \varphi_K^{\tau,0}(q, p), \tau, E + K(q, p, 0) - K(\varphi_K^{0,\tau} \circ \psi_s \circ \varphi_K^{\tau,0}(q, p), \tau)).$$

Note that  $F_0 = \text{Id}_{T^*\mathcal{M}_2}$  and that  $(F_s)$  is the Hamiltonian isotopy associated to the Hamiltonian  $(q, p, \tau, E) \mapsto h(\varphi_K^{\tau,0}(q, p), s)$ .

As  $F_s(\mathcal{G}_0) = \mathcal{G}_s$  and  $\mathcal{G}_1 = \mathcal{G}$ ,  $\mathcal{G}$  is Hamiltonianly isotopic to  $\mathcal{G}_0$ .

Let us now prove that  $\mathcal{G}_0$  is Hamiltonianly isotopic to the zero section  $\mathcal{Z}_{T^*\mathcal{M}_2}$  of  $T^*\mathcal{M}_2$ .

**DEFINITION.** A diffeomorphism  $G : T^*M \rightarrow T^*M$  is *exact symplectic* if  $G^*\lambda - \lambda$  is exact as a 1-form.

**Lemma 1** *Let  $(G_s)$  be an isotopy of exact symplectic diffeomorphisms. Then it is a Hamiltonian isotopy.*

*Proof* If we denote the Liouville 1-form on  $T^*\mathcal{M}_2$  by  $\Theta$ , we have:  $G_s^*\Theta - \Theta = dS_s$  and then if  $X_s$  is the vector field associated to  $(G_s)$ , we have:  $G_s^*(L_{X_s}\Theta) = d\dot{S}_s$ , i.e.  $G_s^*(i_{X_s}d\Theta) + G_s^*(d(i_{X_s}\Theta)) = d\dot{S}_s$ . We finally obtain  $G_s^*(i_{X_s}\Omega) = d(G_s^*(i_{X_s}\Theta) - \dot{S}_s)$ . Hence  $(G_s)$  is Hamiltonian and the associated Hamiltonian is  $i_{X_s}\Theta - \dot{S}_s \circ G_s^{-1}$ .

In order to use Lemma 1, we define  $G_s : T^*\mathcal{M}_2 \rightarrow T^*\mathcal{M}_2$  by:

$$G_s(q, p, \tau, E) = (\varphi_{sK}^{0,\tau}(q, p), \tau, E + s(K(q, p, 0) - K(\varphi_{sK}^{0,\tau}(q, p), \tau))).$$

Then  $G_0 = Id_{T^*\mathcal{M}_2}$  and  $G_1(\mathcal{L}_{T^*\mathcal{M}_2}) = \mathcal{G}_0$ . If we succeed in proving that every  $G_s$  is exact symplectic, we can deduce that  $\mathcal{G}_0$  is Hamiltonianly isotopic to the zero section and hence that  $\mathcal{G}$  is isotopic to the zero section. We prove that only for  $s = 1$  (we can replace  $K$  by  $sK$ ). We use the notations  $G = G_1$ ,  $\varphi_\tau = \varphi_K^{0,\tau}$  and  $K_\tau(q, p) = K(q, p, \tau)$ .

As  $(\varphi_\tau)$  is a Hamiltonian isotopy, every  $\varphi_\tau$  is exact symplectic. We write:  $\varphi_\tau^*\lambda - \lambda = dS_\tau$ . We saw in the proof of Lemma 1 that  $d(i_{X_K}\lambda - K_\tau) = d(\dot{S}_\tau \circ \varphi_\tau^{-1})$ . By adding to  $S_\tau$  a function depending only on  $\tau$ , we can assume that  $i_{X_K}\lambda - K_\tau = \dot{S}_\tau \circ \varphi_\tau^{-1}$ .

Now we compute

$$G^*\Theta - \Theta = \varphi_\tau^*(\lambda) + \varphi_\tau^*(i_{X_K}\lambda)d\tau - \lambda + (K(q, p, 0) - K(\varphi_\tau(q, p), \tau))d\tau.$$

Note that  $K(q, p, 0) = 0$ . We obtain then

$$G^*\Theta - \Theta = dS_\tau + \varphi_\tau^*(i_{X_K}\lambda)d\tau - K(\varphi_\tau(q, p), \tau)d\tau,$$

i.e.  $G^*\Theta - \Theta = dS_\tau + \dot{S}_\tau d\tau$ , then  $G$  is exact symplectic.

## 2.2 The generating function

There is a classical way of quantifying the Lagrangian submanifolds of a cotangent bundle that are Hamiltonianly isotopic to the zero section. This is done by using the so-called generating functions.

The facts that we recall here come from different articles; more precisely, the existence theorem can be found in [22] and [8] and the uniqueness theorem is proved in [24] and [23].

DEFINITIONS.

- Let  $p : E \rightarrow \mathcal{M}_2$  be a finite-dimensional vector bundle over  $\mathcal{M}_2$ . A  $C^2$  function  $S : E \rightarrow \mathbb{R}$  is a *generating function* if its differential  $dS$  is transversal to the manifold  $\mathcal{W} = \{\xi \in T_e^*E; e \in E \text{ and } \xi = 0 \text{ on } T_e(p^{-1}(p(e)))\}$ .
- Then the *critical locus*  $\Sigma_S$  of  $S$  is the set  $\Sigma_S = dS^{-1}(\mathcal{W})$ .
- The map  $i_S : \Sigma_S \rightarrow T^*\mathcal{M}_2$  is defined by  $i_S(e) : T_{p(e)}^*\mathcal{M}_2 \rightarrow T^*\mathcal{M}_2$ ,  $i_S(e)\delta x = dS(e) \cdot \delta e$  where  $\delta e \in T_e E$  is any vector so that  $dp(e) \cdot \delta e = \delta x$ .
- If  $\mathcal{G}$  is a Lagrangian submanifold of  $T^*M$ ,  $S : E \rightarrow \mathbb{R}$  *generates*  $\mathcal{G}$  if  $i_S$  is a diffeomorphism from  $\Sigma_S$  onto  $\mathcal{G}$ .

- When the bundle  $E = \mathcal{M}_2 \times \mathbb{R}^k$  is trivial and there exists a non-degenerate quadratic form  $Q : \mathbb{R}^k \rightarrow \mathbb{R}$  such that  $S = Q$  outside a compact subset, we say that  $S$  is *special*. The *index* of  $S$  is then the index of  $Q$

**Sikorav's existence theorem** *Let  $\mathcal{G}$  be a Lagrangian submanifold of  $T^*\mathcal{M}_2$  that is Hamiltonianly isotopic to the zero section. Then  $\mathcal{G}$  admits a special generating function.*

NOTATION. We denote a special generating function of  $\mathcal{G}$  by  $S(q, \tau; \xi)$  with  $\xi \in \mathbb{R}^k$ .

REMARKS. When  $S : \mathcal{M}_2 \times \mathbb{R}^k \rightarrow \mathbb{R}$  is special, we have:

$$\Sigma_S = \{(q, \xi) \in \mathcal{M}_2 \times \mathbb{R}^k; \frac{\partial S}{\partial \xi}(q, \xi) = 0\};$$

and

$$\forall (q, \xi) \in \Sigma_S, i_S(q, \xi) = (q, \frac{\partial S}{\partial q}(q, \xi)).$$

Observe that the condition that  $S$  is a generating function means that the map  $\frac{\partial S}{\partial \xi}$  is a submersion at every point of  $\Sigma_S$ .

**Proposition 2** *The special functions  $S_0 : (q, \xi) \in M \times \mathbb{R}^k \rightarrow S(q, 0; \xi)$  and  $S_1 : (q, \xi) \rightarrow S(q, 1; \xi)$  generate the Lagrangian submanifold  $\mathcal{L} \subset T^*M$ .*

*Proof* The only non trivial thing to be proved is that the functions  $S_i$  are generating function. Then the fact that they are special and that they generate  $\mathcal{L}$  is straightforward.

We recall that

$$\mathcal{G} = \{\phi'_{\mathcal{X}}(q, p, 0, 0), (q, p) \in \mathcal{L}, t \in [0, 2]\}.$$

Hence if  $(q, p) \in \mathcal{L}$ , we have the equalities  $X_{\mathcal{X}}(q, p, 0, 0) = (0, 0, 1, 0) \in T_{(q, p, 0, 0)}\mathcal{G}$  and  $X_{\mathcal{X}}(\phi_{\mathcal{X}}^{0,1}(q, p), 1, 0) = (0, 0, 1, 0) \in T_{\phi_{\mathcal{X}}^1(q, p, 0, 0)}\mathcal{G}$  because  $\dot{\eta}(0) = \dot{\eta}(1) = \dot{\eta}(1) = 0$ .

Let us recall that  $i_S(q, \tau, \xi) = (q, \frac{\partial S}{\partial q}(q, \tau, \xi), \tau, \frac{\partial S}{\partial \xi}(q, \tau, \xi))$ . Then, for  $j = 0, 1$ , we have  $(D(i_S)^{-1}(q, p, j, 0))(0, 0, 1, 0) = (0, 1, \delta \xi_j) \in T_{i_S^{-1}(q, p, j, 0)}\Sigma_S$ .

As the equation of  $\Sigma_S$  is  $\frac{\partial S}{\partial \xi}(q, \tau, \xi) = 0$ , we deduce that

$$\frac{\partial^2 S}{\partial \tau \partial \xi}(i_S^{-1}(q, p, j)) = -\frac{\partial^2 S}{\partial \xi^2}(i_S^{-1}(q, p, j))\delta \xi_j.$$

This equality implies that for every  $(q, p) \in \mathcal{L}$  and  $j = 0, 1$ , we have

$$\text{Im}D\left(\frac{\partial S}{\partial \xi}\right)(i_S^{-1}(q, p, j, 0)) = \text{Im}D\left(\frac{\partial S}{\partial \xi}\right)(i_S^{-1}(q, p, j, 0))|_{\delta \tau=0} = \mathbb{R}^k,$$

i.e. that  $S_j$  is a generating function.

In the next subsection, we will build what is called a graph selector and we will prove that it doesn't depend on the generating function that we choose. To do that, we need a uniqueness result for the generating functions that is due to C. Viterbo. Let us explain this.

DEFINITIONS. Let  $p : E \rightarrow \mathcal{M}_2$  be a finite dimensional vector bundle and let  $S : E \rightarrow \mathbb{R}$  be a generating function. Let us define the *basic operations* on generating functions:

- *Translation.* If  $c \in \mathbb{R}$ , then  $S' = S + c : E \rightarrow \mathbb{R}$ .
- *Diffeomorphism.* If  $p' : E' \rightarrow \mathcal{M}_2$  is another vector bundle and  $F : E' \rightarrow E$  is a diffeomorphism such that  $p \circ F = p'$ , then  $S' = S \circ F : E' \rightarrow \mathbb{R}$ .
- *Stabilization.* If  $p' : E' \rightarrow \mathcal{M}_2$  is another finite dimensional vector bundle endowed with a function  $Q' : E' \rightarrow \mathbb{R}$  that is quadratic non-degenerate when restricted to the fibers of  $p'$ , then  $S' = S \oplus Q' : E \oplus E' \rightarrow \mathbb{R}$ .

Then, two generating functions are *equivalent* if they can be made equal after a succession of basic operations.

REMARK. Observe that the basic operations that are given by the two first item are reversible in the following sense: if  $S'$  is obtained from  $S$  by such an operation, then  $S$  is obtained from  $S'$  by a similar operation. This is not the case for the third basic operation, for which we can only add variables.

That is why the definition of equivalence is a little subtle:  $S$  is equivalent to  $S'$  if there exists a third generating function  $S''$  so that  $S''$  can be deduced from  $S$  by some basic operations and  $S''$  can be deduced from  $S'$  by some basic operations.

**Viterbo's uniqueness theorem** *Two special functions that generate the same Lagrangian submanifold are equivalent.*

REMARK. The property of being special is not preserved by the basic operations.

### 2.3 Graph selector

Using the generating function  $S$ , we will construct a *graph selector*  $u : \mathcal{M}_2 \rightarrow \mathbb{R}$ . Such a graph selector was introduced by M. Chaperon in [9] (see [20] and [21] too) by using the homology. Here we prefer to use the cohomological approach. We now explain this.

NOTATIONS. Let  $p : E \rightarrow \mathcal{M}_2$  be a finite dimensional vector bundle. If  $S : E \rightarrow \mathbb{R}$  is a function that generates a Lagrangian submanifold,  $q \in \mathcal{M}_2$  and  $a \in \mathbb{R}$  is a real number, we denote the sublevel with height  $a$  at  $q$  by

$$S_q^a = \{e \in E; \quad p(e) = q \quad \text{and} \quad S(e) \leq a\}$$

and we use the notation  $S_q = S|_{E_q}$ .

When  $S$  is special with index  $m$ , there exists  $N \geq 0$  such that all the critical values are in  $(-N, N)$ . Then the De Rham relative cohomology space with compact support  $H^*(E_q, S_q^{-N})$  is isomorphic to  $\mathbb{R}$  for  $*$  =  $m$  and trivial if  $*$   $\neq m$ . We denote by  $\alpha_q$  a closed  $m$ -form with compact support on  $E_q$  such that  $\alpha_q|_{S_q^{-N}} = 0$  and  $0 \neq [\alpha_q] \in H^m(E_q, S_q^{-N})$ .

If  $a \in (-N, N)$ , we use the notation  $i_a : (S_q^a, S_q^{-N}) \rightarrow (E_q, S_q^{-N})$  for the inclusion and then  $i_a^* : H^m(E_q, S_q^{-N}) \rightarrow H^m(S_q^a, S_q^{-N})$ . The graph selector  $u : \mathcal{M}_2 \rightarrow \mathbb{R}$  is then defined by:

$$u(q) = \sup\{a \in \mathbb{R}; [i_a^* \alpha_q] = 0\} = \inf\{a \in \mathbb{R}; [i_a^* \alpha_q] \neq 0\}.$$

**Proposition 3** *Let  $p : E \rightarrow \mathcal{M}_2$  be a finite dimensional vector bundle, let  $S : E \rightarrow \mathbb{R}$  be a special generating function with index  $m$  and let  $\sigma : E' \rightarrow \mathbb{R}$  be a generating function that is got from  $S$  after a succession of basic operations. If there are exactly  $k$  stabilizations among these basic operations (with indices  $m_1, \dots, m_k$ ), the sum of all the indices is denoted by  $\ell = m + \sum_{j=1}^k m_j$ .*

*Then, for  $N$  positive large enough,  $H^\ell(E'_q, \sigma_q^{-N})$  is isomorphic to  $\mathbb{R}$ ; if  $[A_q]$  is one of its generator, we can define a graph selector by*

$$U(q) = \sup\{a \in \mathbb{R}; [i_a^* A_q] = 0\} = \inf\{a \in \mathbb{R}; [i_a^* A_q] \neq 0\}.$$

*This graph selector is equal to the one associated to  $S$  plus a constant.*

*Proof* If the basic operation that we use is a translation or a diffeomorphism, the proposition is straightforward. The only non trivial case concerns stabilization. From now, we forget the translations and the constants and we can assume that we are in the following case.

Assume that  $S : E \rightarrow \mathbb{R}$  is a generating function such that after a fiber diffeomorphism  $\psi : E \rightarrow E$ ,  $S \circ \psi$  is non-degenerate quadratic in every fiber outside some compact subset, the quadratic form being denoted by  $Q_0$  and having index  $m_0$ . Then  $H^{m_0}(E_q, S_q^{-N})$  is isomorphic to  $\mathbb{R}$  and  $H^*(E_q, S_q^{-N}) = \{0\}$  if  $*$   $\neq m_0$ . For such a function we can define a graph selector  $u$  as before (even if this function is not special). Assume that  $Q : F \rightarrow \mathbb{R}$  is a non-degenerate quadratic form with index  $m$  when restricted to the fibers of  $p' : F \rightarrow \mathcal{M}_2$  and let us use the notation  $\mathcal{S} = (S \circ \psi) \oplus Q : E \oplus F \rightarrow \mathbb{R}$ .

*Dimension of  $H^*(E_q \oplus F_q, \mathcal{S}_q^{-N-C})$*

Observe that  $|\mathcal{S} - Q_0 \oplus Q|$  is bounded by some constant  $C$ . Hence we have:

$$\{Q_0 \oplus Q \leq a - C\} \subset \{\mathcal{S} \leq a\} \subset \{Q_0 \oplus Q \leq a + C\}.$$

Then we choose  $N \geq 0$  such that all the critical values of  $S$  (and then of  $\mathcal{S}$ ) are in  $(-N, N)$ . Then the inclusion maps induce the following homomorphisms

$$H^*(E_q \oplus F_q, Q_0 \oplus Q \leq -N) \xrightarrow{J_3^*} H^*(E_q \oplus F_q, \mathcal{S}_q^{-N-C}) \xrightarrow{J_2^*}$$

$$H^*(E_q \oplus F_q, Q_0 \oplus Q \leq -N - 2C) \xrightarrow{j_1^*} H^*(E_q \oplus F_q, \mathcal{S}_q^{-N-3C}).$$

As the pairs  $(E_q \oplus F_q, Q_0 \oplus Q \leq -N)$  and  $(E_q \oplus F_q, Q_0 \oplus Q \leq -N - 2C)$  are homotopically equivalent and as the pairs  $(E_q \oplus F_q, \mathcal{S}_q^{-N-C})$  and  $(E_q \oplus F_q, \mathcal{S}_q^{-N-3C})$  are homotopically equivalent, the maps  $j_1^* \circ j_2^*$  and  $j_2^* \circ j_3^*$  are isomorphisms, and then  $j_2^*$  is an isomorphism too. We deduce that  $H^*(E_q \oplus F_q, \mathcal{S}_q^{-N-C})$  is isomorphic to  $\mathbb{R}$  if  $* = m_0 + m$  and  $\{0\}$  if  $* \neq m_0 + m$ .

The same is true if we replace  $\mathcal{S}$  by the function  $\sigma = S \oplus Q$  that will be denoted by  $\sigma$  from now.

#### *A first inequality between the two graph selectors*

Let  $\varepsilon$  be a positive number. We will prove that  $U(q) \leq u(q) + \varepsilon =: a + \frac{\varepsilon}{2}$ . Let  $\alpha$  be a closed  $m_0$ -form that vanishes on  $S_q^{-N}$  and is such that  $0 \neq [i_a^* \alpha] \in \bar{H}^{m_0}(S_q^a, S_q^{-N})$  and let  $\beta$  be a closed  $m$ -form that vanishes on  $Q_q^\varepsilon$  and such that  $0 \neq [i_{\frac{\varepsilon}{2}}^* \beta] \in H^m(Q_q^{\frac{\varepsilon}{2}}, Q_q^{-\varepsilon})$ .

We denote by  $A$  (resp.  $B$ ) a  $m_0$ -cycle of  $S_q^a$  with boundary in  $S_q^{-N}$  (resp.  $m$ -cycle of  $Q_q^{\frac{\varepsilon}{2}}$  with boundary in  $Q_q^{-\varepsilon}$ ) such that  $\alpha(A) \neq 0$  (resp. such that  $\beta(B) \neq 0$ ). We use the notation  $\mu_A = \sup Q_{|A}$  and  $\mu_B = \sup S_{|B}$ . Using the gradient flow of  $Q_0$  on  $S_q^{-N}$  to push  $A$  or the gradient flow of  $Q$  on  $Q_q^{-\varepsilon}$  to push  $B$ , we can assume that  $S_{|\partial A} \leq -\varepsilon - N - \mu_A$  and  $Q_{|\partial B} \leq -\varepsilon - N - \mu_B$ ; observe that this implies that  $\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B) \subset \sigma^{-\varepsilon-N}$ .

Then the cup product  $\alpha \vee \beta$  is a closed  $(m + m_0)$ -form that vanishes in  $(Q_q^{-\varepsilon} \times F_q) \cup (E_q \times S_q^{-N})$  and such that  $(\alpha \vee \beta)(A \times B) \neq 0$ . As the set  $(Q_q^{-\varepsilon} \times F_q) \cup (E_q \times S_q^{-N})$  contains  $\sigma^{-\varepsilon-N}$  and as the support of  $A \times B$  is in  $S_q^a \times Q_q^{\frac{\varepsilon}{2}} \subset \sigma^{a+\frac{\varepsilon}{2}}$ , we deduce that  $0 \neq [i_{a+\frac{\varepsilon}{2}}^* (\alpha \vee \beta)] \in H^{m+m_0}(\sigma^{a+\frac{\varepsilon}{2}}, \sigma^{-\varepsilon-N})$  and thus  $U(q) \leq a + \frac{\varepsilon}{2} = u(q) + \varepsilon$ . Hence we have  $U(q) \leq u(q)$ .

#### *The reverse inequality between the two graph selectors*

Let us now prove that for  $\varepsilon > 0$ , we have  $U(q) \geq u(q) - \varepsilon = a - \frac{\varepsilon}{2}$ . We use the notation  $j : S_q^{a-\frac{\varepsilon}{2}} \rightarrow E_q \oplus F_q$ ,  $j_1 : S_q^a \rightarrow E_q$  and  $j_2 : Q_q^{-\varepsilon} \rightarrow F_q$  for the inclusion maps. As  $H^*(S_q^a, S_q^{-N})$  and  $H^*(Q_q^{-\frac{\varepsilon}{2}}, Q_q^{-\varepsilon})$  are trivial, there exists a  $(m_0 - 1)$ -form  $\alpha_1$  on  $S_q^a$  such that  $\alpha_1|_{S_{\leq -N}} = 0$  and  $j_1^* \alpha = d\alpha_1$  and a  $(m - 1)$ -form  $\beta_1$  on  $Q_q^{-\frac{\varepsilon}{2}}$  such that  $\beta_1|_{Q_{\leq -\varepsilon}} = 0$  and  $j_2^* \beta = d\beta_1$ .

Observe that  $(S_q^a \times Q_q^{-\varepsilon}, S_q^{-N} \times Q_q^{-\frac{\varepsilon}{2}})$  is an excisive couple (see [17]), hence all the following cohomology spaces vanish because they can be expressed with the trivial spaces  $H^*(S_q^a, S_q^{-N})$  and  $H^*(Q_q^{-\frac{\varepsilon}{2}}, Q_q^{-\varepsilon})$

$$H^*(S_q^a \times Q_q^{-\frac{\varepsilon}{2}}, (S_q^a \times Q_q^{-\varepsilon}) \cup (S_q^{-N} \times Q_q^{-\frac{\varepsilon}{2}})) = \{0\}.$$

We have  $d(\alpha_1 \vee \beta_1) = d((-1)^{m_0} \alpha \vee \beta_1) = \alpha \vee \beta$ . We deduce that there exists a  $(m_0 + m - 2)$ -form  $\mu$  on  $S_q^a \times Q_q^{-\frac{\varepsilon}{2}}$  that vanishes on  $(S_q^a \times Q_q^{-\varepsilon}) \cup (S_q^{-N} \times Q_q^{-\frac{\varepsilon}{2}})$  and is such

that  $\alpha_1 \vee \beta - (-1)^{m_0} \alpha \vee \beta_1 = d\mu$ . We can extend  $\mu$  in a  $(m_0 + m - 2)$ -form that is defined on  $E_q \oplus F_q$  and vanishes on  $(E_q \times Q_q^{-\varepsilon}) \cup (S_q^{-N} \times F_q)$ . Then the  $(m_0 + m - 1)$ -form  $\alpha_1 \vee \beta$  that is defined on  $S_q^a \times F_q$  coincides on the intersection of the two sets with the  $(m_0 + m - 1)$ -form  $(-1)^{m_0} \alpha \vee \beta_1 + d\mu$  that is defined on  $E_q \times Q_q^{-\frac{\varepsilon}{2}}$ . Together, they define a  $(m_0 + m - 1)$ -form  $\mu_1$  on  $(S_q^a \times F_q) \cup (E_q \times Q_q^{-\frac{\varepsilon}{2}}) \supset \sigma_q^{a-\frac{\varepsilon}{2}}$  such that

- $\mu_1$  vanishes on  $(E_q \times Q_q^{-\varepsilon}) \cup (S_q^{-N} \times F_q) \supset \sigma_q^{-N-\varepsilon}$ ;
- $d\mu_1 = \alpha \vee \beta$ .

We deduce that  $0 = j^*(\alpha \vee \beta) \in H^{m+m_0}(\sigma_q^{u(q)-\varepsilon}, \sigma_q^{-N-\varepsilon})$  and then that  $U(q) \geq u(q) - \varepsilon$ . Hence  $U(q) \geq u(q)$  and finally  $u(q) = U(q)$ .

NOTATIONS. From now we denote by  $S : (q, \tau, \xi) \in E \rightarrow S(q, \tau, \xi)$  a special generating function for  $\mathcal{G}$ . The critical locus is denoted by  $\Sigma$  and the associated embedding is  $i = i_S : \Sigma \rightarrow T^*\mathcal{M}_2$ . We denote by  $u : (q, \tau) \in \mathcal{M}_2 \rightarrow u(q, \tau)$  the graph selector associated to  $S$ .

Following the proofs that are contained in [20] or [21] for the homology, we will prove

**Proposition 4** *Let  $u : \mathcal{M}_2 \rightarrow \mathbb{R}$  be a graph selector for the special generating function  $S : \mathcal{M}_2 \times \mathbb{R}^k \rightarrow \mathbb{R}$ . Then  $u$  is a Lipschitz function that is  $C^1$  on an open subset  $U_0 \subset \mathcal{M}_2$  with full Lebesgue measure, and for every  $z = (q, t) \in U_0$ , the following properties hold*

$$J_u(z) \in \mathcal{G}, \quad \text{and} \quad u(z) = S \circ i^{-1}(J_u(z)), \quad (4)$$

where  $J_u(z) = (q, d_q u(z), t, \frac{\partial u}{\partial t}(z)) \in T^*M \times \mathbb{T}_2 \times \mathbb{R}$  and with the usual identification  $T^*\mathcal{M}_2 = T^*M \times \mathbb{T}_2 \times \mathbb{R}$ .

*Proof* We assume that  $S = Q$  on all the levels that are not in  $(-N, N)$  and we denote the index of  $Q$  by  $m$ . Let us fix  $z \in \mathcal{M}$ . We denote by  $\alpha$  a  $m$ -form on  $\mathbb{R}^k$  that vanishes on  $Q^{-N}$  and is such that  $0 \neq [\alpha] \in H^m(\mathbb{R}^k, Q^{-N})$ . Because there is a change in the topology of the sublevel with height  $u(z)$ ,  $u(z)$  is a critical value of  $S_z$ .

Let us prove that  $u$  is Lipschitz. Observe that the function  $\mathcal{V} : \mathcal{M} \times \mathcal{M} \times \mathbb{R}^k \rightarrow \mathbb{R}$  that is defined by  $\mathcal{V}(z, z', \xi) = S(z', \xi) - S(z, \xi)$  is  $C^1$  and has compact support. Hence there exists a constant  $L > 0$  such that

$$\forall z, z' \in \mathcal{M}, \forall \xi \in \mathbb{R}^k, |S(z, \xi) - S(z', \xi)| \leq L.d(z, z').$$

We deduce that for every  $a \in \mathbb{R}$ ,  $S_z^a \subset S_{z'}^{a+L.d(z, z')}$ . Then the inclusion maps induce the following maps (note that  $S_z^{-N} = S_{z'}^{-N}$ ):

$$H^m(\mathbb{R}^k, S_z^{-N}) \xrightarrow{j_2^*} H^m(S_{z'}^{u(z)+L.d(z, z')+\varepsilon}, S_{z'}^{-N}) \xrightarrow{J_1^*} H^m(S_z^{u(z)+\varepsilon}, S_z^{-N}).$$

We know that  $0 \neq (j_2 \circ j_1)^* \alpha \in H^m(S_z^{u(z)+\varepsilon}, S_z^{-N})$ . This implies that  $j_2^* \alpha \neq 0$  and then that  $u(z') \leq u(z) + \varepsilon + L.d(z, z')$ . This is also valid when we exchange  $z$  and  $z'$ . Therefore, when we let  $\varepsilon$  go to zero, we get :

$$|u(z) - u(z')| \leq L.d(z, z').$$

Let us now prove that there exists an open subset  $U_0$  of  $\mathcal{M}$  with full Lebesgue measure on which  $u$  is  $C^1$ . Observe that the set  $U_1$  of the  $z \in \mathcal{M}$  where  $S_z$  is Morse is exactly the set of regular values of the restriction to  $\Sigma_S$  of the first projection  $(z, v) \in \mathcal{M} \times \mathbb{R}^k \mapsto z$  and then has full Lebesgue measure by Sard's theorem. It is open. We denote by  $U_0$  the set of the  $z \in U_1$  such that the critical points of  $S_z$  have pairwise distinct critical values. Let us prove that  $U_1 \setminus U_0$  has only isolated points: this will imply that  $U_0$  is open and has full Lebesgue measure. Let us consider  $z \in U_1 \setminus U_0$ . As  $S_z$  is Morse,  $\Sigma_S$  is transverse to  $\{z\} \times \mathbb{R}^k$  and then above a neighbourhood  $V_z$  of  $z$  in  $\mathcal{M}$ ,  $\Sigma_S$  is the union of  $j$  graphs, the graphs of  $\eta_1, \dots, \eta_j : V_z \rightarrow \mathbb{R}^k$ . If we use the notation  $\psi_j(z') = \frac{\partial S}{\partial q}(z', \eta_j(z'))$ , then  $\mathcal{G}$  is the union of the disjoint graphs of  $\psi_1, \dots, \psi_j$  above  $V_z$ . For  $z' \in V_z$ ,  $u(z')$  is a critical value of  $S_{z'}$  and then is one of the real numbers  $S(z', \eta_1(z')), \dots, S(z', \eta_j(z'))$ . Note that every  $S(\cdot, \eta_i(\cdot))$  is  $C^1$  and that  $\frac{\partial S(\cdot, \eta_i(\cdot))}{\partial z} = \psi_i$ . As the  $\psi_i(z')$  are pairwise distinct, for  $i \neq j$ ,  $\{S(\cdot, \eta_i(\cdot)) = S(\cdot, \eta_j(\cdot))\}$  has only isolated points.

Let us now consider  $z \in U_0$ . We can define a connected neighbourhood  $V_z$ ,  $\eta_1, \dots, \eta_j$  and  $\psi_1, \dots, \psi_j$  exactly as before. Then every  $u(z')$  is one of the  $S(z', \eta_i(z'))$ . Because  $V_z$  is a connected part of  $U_0$ , there exists exactly one  $i$  such that  $\forall z' \in V_z, u(z') = S(z', \eta_i(z'))$ . Then we have  $du(z') = \frac{\partial S}{\partial z}(z', \eta_i(z')) = \psi_i(z')$  and we deduce that  $u(z') = S \circ i_S^{-1}(J_u(z'))$  and  $J_u(z') \in \mathcal{G}$ .

**Proposition 5** *There exist a real constant  $c$  such that the following identity holds*

$$\forall q \in M, \quad u(q, 1) = u(q, 0) - c \quad (5)$$

*Proof* We proved in Proposition 2 that  $S(q, 0; \xi)$  and  $S(q, 1; \xi)$  are two generating functions for  $\mathcal{L}$ . We deduce from Proposition 3 the wanted result.

**Corollary 3** *For the same constant  $c$  that is defined in Proposition 5, the function  $S \circ i_S^{-1}$  satisfies the identity*

$$S \circ i_S^{-1}(q, p, 1, 0) = S \circ i_S^{-1}(q, p, 0, 0) - c, \quad (q, p) \in \mathcal{L}. \quad (6)$$

*Proof* As  $S(q, 0; \xi)$  and  $S(q, 1; \xi)$  are two generating functions for  $\mathcal{L}$ , the functions  $(q, p) \in \mathcal{L} \mapsto S \circ i^{-1}(q, p, 1, 0)$  and  $(q, p) \in \mathcal{L} \mapsto S \circ i^{-1}(q, p, 0, 0)$  are two primitive on  $\mathcal{L}$  of the Liouville 1-form  $\lambda$ . Hence their difference is a constant.

Moreover,  $u(\cdot, 0)$  and  $u(\cdot, 1)$  are two graph selectors for  $\mathcal{L}$  so that  $u(\cdot, 0) - u(\cdot, 1) = c$ . Hence there exists a dense open subset  $V_0$  of  $M$  with full Lebesgue measure such that for  $i = 1, 2$

$$\forall q \in V_0, (q, d_q u(q, 0)) = (q, d_q u(q, 1)) \in \mathcal{L} \quad \text{and} \quad u(q, i) = S \circ i_S^{-1}(q, d_q u(q, i), i, 0).$$

Take  $q \in V_0$ . By Proposition 4, we have for  $(q, p) = (q, d_q u(q, 0)) = (q, d_q u(q, 1)) \in \mathcal{L}$  that

$$S \circ i_S^{-1}(q, p, 1, 0) = u(q, 1) = u(q, 0) - c = S \circ i_S^{-1}(q, p, 0, 0) - c.$$



### 3 Construction of a dominated function.

In this section, we come back to the original problem, and construct what is called a *dominated function* for the Lagrangian  $L$  that is associated to  $H$ , where we recall the definition of the Lagrangian that we gave in the introduction.

NOTATION. The *Lagrangian*  $L : TM \times \mathbb{T} \rightarrow \mathbb{R}$  is the function that is associated to  $H$  via the Legendre duality.

$$\forall (q, v, t) \in TM \times \mathbb{T}, L(q, v, t) = \inf_{p \in T_q^*M} (p \cdot v - H(q, p, t)).$$

We recall that  $L$  is as regular as  $H$  is,  $C^2$ -convex in the fiber direction and super-linear in the fiber direction (see e.g. [11]).

DEFINITION. A function  $U : \mathcal{M}_1 = M \times \mathbb{T} \rightarrow \mathbb{R}$  is *dominated* by  $L + c$  if it is Lipschitz and if for every continuous and piecewise  $C^1$  arc  $\gamma : [a, b] \rightarrow M$ , we have

$$U(\gamma(b), b) - U(\gamma(a), a) \leq \int_a^b (L(\gamma(t), \dot{\gamma}(t), t) + c) dt.$$

The goal of this section is to build a function  $u$  that is dominated by  $L + c$  and to prove some properties for this function.

Then, in the last section, we will prove that  $u$  is everywhere differentiable and that  $\mathcal{L}$  is contained in the graph of  $q \mapsto du(q, 0)$ . After that, we will prove that  $du$  is  $C^1$ .

#### 3.1 Construction of a dominated function.

Let us introduce a notation.

NOTATION. We define  $u : M \times [0, 1] \rightarrow \mathbb{R}$  by  $u(q, t) = u(q, \eta^{-1}(t)) + ct$ .

Observe that a consequence of Proposition 5 is that  $u(\cdot, 0) = u(\cdot, 1)$ . Hence we can consider  $u$  as a function defined on  $\mathcal{M}_1 = M \times \mathbb{T}$ .

**Proposition 6** *The function  $u$  is Lipschitz and dominated by  $L + c$ .*

*Proof* We postpone the proof that  $u$  is Lipschitz after the proof of the domination property, but we use the fact that  $u$  is Lipschitz in the first part of our proof.

*The domination property*

Let  $\gamma : [a, b] \rightarrow M$  be a  $C^1$  arc with  $[a, b] \subset (0, 1)$  and assume that the image of  $t \in [a, b] \mapsto (\gamma \circ \eta(t), t) \in \mathcal{M}_1$  is Lebesgue almost everywhere in  $U_0$  ( $U_0$  was defined in Proposition 4). Then

$$u(\gamma(b), b) - u(\gamma(a), a) = u(\gamma(b), \eta^{-1}(b)) - u(\gamma(a), \eta^{-1}(a)) + c(b - a),$$

and if we use the notation  $\delta = u(\gamma(b), \eta^{-1}(b)) - u(\gamma(a), \eta^{-1}(a))$

$$\delta = \int_{\eta^{-1}(a)}^{\eta^{-1}(b)} \left( d_q u(\gamma(\eta(t)), t) \dot{\gamma}(\eta(t)) \dot{\eta}(t) + \frac{\partial u}{\partial t}(\gamma \circ \eta(t), t) \right) dt.$$

Young inequality for dual convex functions tells us that

$$\forall p \in T_q^* M, \forall v \in T_q M, \forall t \in \mathbb{T}, p \cdot v \leq H(q, p, t) + L(q, v, t).$$

Hence we have  $\delta \leq$

$$\int_{\eta^{-1}(a)}^{\eta^{-1}(b)} \left[ \dot{\eta}(t) (H(\gamma(\eta(t)), d_q u(\gamma(\eta(t)), t), \eta(t)) + L(\gamma(\eta(t)), \dot{\gamma}(\eta(t)), \eta(t))) + \frac{\partial u}{\partial t}(\gamma \circ \eta(t), t) \right] dt.$$

Proposition 4 tells us that  $u$  is a graph selector for  $\mathcal{G}$  above  $U_0$ . We can therefore replace in the integral  $\frac{\partial u}{\partial t}(\gamma \circ \eta(t), t)$  by  $-K(\gamma(\eta(t)), p(t), t) = -\dot{\eta}(t)H(\gamma(\eta(t)), p(t), \eta(t))$ , where we set  $p(t) = d_q u(\gamma(\eta(t)), t)$ . Using a change of variable  $s = \eta(t)$  we obtain

$$\delta \leq \int_{\eta^{-1}(a)}^{\eta^{-1}(b)} \dot{\eta}(t) L(\gamma(\eta(t)), \dot{\gamma}(\eta(t)), \eta(t)) dt = \int_a^b L(\gamma(s), \dot{\gamma}(s), s) ds.$$

This gives the domination property

$$u(\gamma(b), b) - u(\gamma(a), a) \leq \int_a^b (L(\gamma(s), \dot{\gamma}(s), s) + c) ds. \quad (7)$$

How can we conclude for general  $\gamma : [a, b] \rightarrow M$  that are continuous and piecewise  $C^1$ ?

- if  $[a, b] \subset (0, 1)$  and  $\gamma$  is  $C^1$ , by Lemma 2 below applied to  $\gamma \circ \eta$ , we can approximate  $\gamma$  in topology  $C^1$  by a sequence  $(\gamma_n)_n$  such that  $(\gamma_n \circ \eta(t), t) \in U_0$  for almost every  $t \in [\eta^{-1}(a), \eta^{-1}(b)]$ , hence the domination inequality holds for every  $\gamma_n$ . Taking now the limit  $n \rightarrow +\infty$ , we find that inequality (7) holds for our curve  $\gamma$ .
- if  $[a, b] \subset [0, 1]$  and  $\gamma$  is  $C^1$ , we can find a decreasing sequence  $(a_n)$  and an increasing sequence  $(b_n)$  so that  $(a, b) = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ ; then every  $\gamma|_{[a_n, b_n]}$  is dominated and by taking a limit  $\gamma$  is dominated;
- for general  $\gamma$ , we can cut  $\gamma$  in sub-arcs  $\gamma_1, \dots, \gamma_n$  that are  $C^1$  and defined on some intervals  $I_k$  that are contained in some intervals  $[n_k, n_k + 1]$  with  $n_k \in \mathbb{Z}$ ; then we have the domination property for every  $\gamma_j$  and hence for their concatenation  $\gamma = \gamma_1 * \dots * \gamma_n$ .

**Lemma 2** *Given an interval  $[\alpha, \beta] \subset (0, 1)$ , a set of full measure  $U_0 \subset \mathcal{M}_1$  and a  $C^1$  curve  $\tau : [\alpha, \beta] \rightarrow M$ , there exists a sequence of  $C^1$  curves  $\tau_n : [\alpha, \beta] \rightarrow M$ ,  $n \in \mathbb{N}$  such that  $(\tau_n)_{n \in \mathbb{N}}$  converges to  $\tau$  in the  $C^1$ -topology, and for every  $n \in \mathbb{N}$ ,  $(\tau_n(t), t) \in U_0$  for almost every  $t \in [\alpha, \beta]$ .*

*Proof* Without loss of generality, we can assume that  $\tau$  is defined in a slightly bigger interval  $[\alpha', \beta'] \subset (0, 1)$  such that  $[\alpha, \beta] \subset (\alpha', \beta')$ . The curve  $\sigma : [\alpha', \beta'] \rightarrow \mathcal{M}_1$ ,  $\sigma(t) = (\tau(t), t)$  is a  $C^1$ -embedding, and it can be embedded in a tubular neighbourhood, that is to say, there exist a  $C^1$ -embedding  $\Lambda : [\alpha', \beta'] \times \mathcal{O} \rightarrow \mathcal{M}_1$ ,  $\Lambda(t, \xi) = (\Gamma(t, \xi), T(t, \xi))$  such that  $\Gamma(t, 0) = \tau(t)$  and  $T(t, 0) = t$ , where  $\mathcal{O}$  is an open neighbourhood of 0 in  $\mathbb{R}^n$ . Let us prove that it is always possible to find a tubular neighbourhood  $\tilde{\Lambda}$  of  $\sigma$  so that  $\tilde{\Lambda}(t, \xi) = (\tilde{\Gamma}(t, \xi), t)$ . Indeed, let  $F : [\alpha', \beta'] \times \mathcal{O} \rightarrow \mathbb{R} \times \mathcal{O}$  be the map defined by  $F(t, \xi) = (T(t, \xi), \xi)$ . Since  $T(t, 0) = t$ , the differential  $D_{(t,0)}F$  is the identity. Eventually shrinking  $\mathcal{O}$  we can assume that  $D_{(t,\xi)}F$  is invertible for every  $(t, \xi) \in [\alpha', \beta'] \times \mathcal{O}$ . Since the map  $t \mapsto T(t, \xi)$  is  $C^1$ -close to the identity for  $\xi$  sufficiently small, hence injective, we can also assume that  $F$  is injective, and therefore  $F$  defines a diffeomorphism from  $[\alpha', \beta'] \times \mathcal{O}$  to a neighbourhood of  $[\alpha, \beta] \times \{0\}$ . By definition of  $F$ , if we set  $\tilde{\Lambda} = \Lambda \circ F^{-1}$  we get  $\tilde{\Lambda}(t, \xi) = (\tilde{\Gamma}(t, \xi), t)$ , where  $\tilde{\Gamma} = \Gamma \circ F^{-1}$ . Now  $F([\alpha', \beta'] \times \mathcal{O})$  is a neighbourhood of  $[\alpha, \beta] \times \{0\}$ , hence we can find an open neighbourhood of 0 in  $\mathbb{R}^n$ , here denoted  $\tilde{\mathcal{O}}$ , such that  $[\alpha, \beta] \times \tilde{\mathcal{O}} \subset F([\alpha', \beta'] \times \mathcal{O})$ . Since  $\tilde{\Lambda}$  is a  $C^1$ -diffeomorphism, the set  $V_0 = \tilde{\Lambda}^{-1}(U_0 \cap \tilde{\Lambda}([\alpha, \beta] \times \tilde{\mathcal{O}}))$  has full measure in  $[\alpha, \beta] \times \tilde{\mathcal{O}}$ , and by Fubini Theorem, for almost every  $\xi \in \tilde{\mathcal{O}}$ , the set of  $t \in [\alpha, \beta]$  such that  $(t, \xi) \in V_0$  has full measure in  $[\alpha, \beta]$ , therefore, we can find a sequence  $(\xi_n)_n$  in  $\tilde{\mathcal{O}}$  such that  $\xi_n \rightarrow 0$  and for almost every  $t \in [\alpha, \beta]$  we have  $(\tilde{\Gamma}(t, \xi_n), t) \in U_0$ . By defining  $\tau_n(t) = \tilde{\Gamma}(t, \xi_n)$  we have the desired property.

#### The Lipschitz property

Let us remark that  $\eta^{-1} : [0, 1] \rightarrow [0, 1]$  is an absolutely continuous function. Indeed, it is a  $C^\infty$  function on the open interval  $(0, 1)$ , and if we set  $g(t) = (\eta^{-1})'(t)$  for  $t \in (0, 1)$ , for every segment  $[a, b] \subset (0, 1)$  we have

$$\eta^{-1}(b) - \eta^{-1}(a) = \int_a^b g(t) dt, \quad (8)$$

and by construction of  $\eta$  we know that  $g(t) > 0$  for  $t \in (0, 1)$ . By continuity of  $\eta^{-1}$ , if we take the limits  $a \rightarrow 0$  and  $b \rightarrow 1$ , we find that  $g$  is absolutely integrable on  $(0, 1)$ , and identity (8) holds for every  $[a, b] \subset [0, 1]$ , hence  $\eta^{-1}$  is absolutely continuous. As  $u$  is Lipschitz, the function  $u$  that we defined by  $u(q, t) = u(q, \eta^{-1}(t)) + ct$  is (uniformly) absolutely continuous in the  $t$ -direction and (uniformly) Lipschitz in the  $q$  direction. Hence, to prove that  $u$  is Lipschitz, we just have to prove that its derivative, which is defined Lebesgue almost everywhere, is bounded on a set with full Lebesgue measure.

Observe that for every segment  $[a, b] \subset (0, 1)$ , the map  $\eta|_{[a,b]} : [a, b] \rightarrow [\eta(a), \eta(b)]$  is a bi-lipschitz homeomorphism; we deduce that the set

$$\mathcal{U}_0 = \{(q, \eta(t)); (q, t) \in U_0 \cap (M \times (0, 1))\} \quad (9)$$

has full Lebesgue measure in  $\mathcal{M}_1 = M \times \mathbb{T}$ . For  $(q, t) \in \mathcal{U}_0$ , we have  $(q, \eta^{-1}(t)) \in U_0$  and then  $(q, \eta^{-1}(t), du(q, \eta^{-1}(t))) \in \mathcal{G}$ . This implies that  $d_q u(q, \eta^{-1}(t))$  is (uniformly) bounded on  $\mathcal{U}_0$  and

$$\frac{\partial u}{\partial t}(q, \eta^{-1}(t)) = -K(q, d_q u(q, \eta^{-1}(t)), t) = -\dot{\eta}(\eta^{-1}(t))H(q, d_q u(q, \eta^{-1}(t)), t).$$

We deduce that

$$du(q,t)(\delta q, \delta t) = d_{qu}(q, \eta^{-1}(t))\delta q + \frac{1}{\dot{\eta}(\eta^{-1}(t))} \frac{\partial u}{\partial t}(q, \eta^{-1}(t))\delta t \quad (10)$$

is equal to

$$du(q,t)(\delta q, \delta t) = d_{qu}(q, \eta^{-1}(t))\delta q - H(q, d_{qu}(q, \eta^{-1}(t)), t)\delta t$$

and thus  $du$  is bounded above  $\mathcal{U}_0$ .

Let us now conclude. Given now two points  $(q, t)$  and  $(q', t') \in \mathcal{M}_1$ , we have

$$\begin{aligned} |u(q', t') - u(q, t)| &\leq |u(q', t') - u(q, t')| + |u(q, t') - u(q, t)| \\ &\leq A \text{dist}(q', q) + |u(q, t') - u(q, t)| \end{aligned} \quad (11)$$

where  $\text{dist}(\cdot, \cdot)$  is a Riemannian distance on  $M$ ,  $A$  is a positive constant independent from  $(q, t)$  and  $(q', t')$ . By an argument similar to the one given in proof of Lemma 2, and eventually cutting the segment  $s \mapsto (q, s)$  in a finite number of pieces, we can find a sequence of points  $(q_n)_n$  in  $M$  converging to  $q$  and such that for every  $n \in \mathbb{N}$ , the point  $(q_n, s)$  is in  $\mathcal{U}_0$  for almost every  $s \in [t, t']$  (without loss of generality we assume  $t < t'$ ). Since we know that  $du$  is bounded above  $\mathcal{U}_0$  we find

$$|u(q_n, t') - u(q_n, t)| \leq \int_t^{t'} \left| \frac{\partial u}{\partial t}(q_n, s) \right| ds \leq B|t' - t|,$$

for some constant  $B > 0$ . Taking now the limit  $n \rightarrow +\infty$  and replacing in (11) we finish the proof.

### 3.2 The dominated function $u$ can be seen as a kind of graph selector

In this part, we construct an extended Hamiltonian of  $H$  and an extended Lagrangian submanifold  $\mathcal{Y}$  of  $\mathcal{L}$  by using  $H$ . We will prove that in some sense,  $u$  is a graph selector for  $\mathcal{Y}$ .

NOTATION. We introduce the autonomous Hamiltonian  $\mathcal{H}$  on  $T^*\mathcal{M}_1 = T^*M \times \mathbb{T}_1 \times \mathbb{R}$  that is defined by

$$\mathcal{H}(q, p, \tau, e) = H(q, p, \tau) + e.$$

The Hamiltonian equations for  $\mathcal{H}$  are

$$\begin{cases} \frac{dq}{dt} = \frac{\partial H}{\partial p}(q, p, \tau), & \frac{d\tau}{dt} = 1, \\ \frac{dp}{dt} = -\frac{\partial H}{\partial q}(q, p, \tau), & \frac{de}{dt} = -\frac{\partial H}{\partial \tau}(q, p, \tau), \end{cases} \quad (12)$$

and the flow of (12) is given by

$$\phi_{\mathcal{H}}^t(q, p, \tau, e) = (\phi_H^{\tau, \tau+t}(q, p), \tau + t, e + H(q, p, \tau) - H(\phi_H^{\tau, \tau+t}(q, p), \tau + t)). \quad (13)$$

If we denote by  $F_E(q, p, \tau, e) = (q, p, \tau, e + E)$  the translation in the energy direction by  $E$ , observe that  $F_E \circ \phi_{\mathcal{H}}^t = \phi_{\mathcal{H}}^t \circ F_E$ . Hence the restriction of  $(\phi_{\mathcal{H}}^t)$  to every level  $\{\mathcal{H} = E\}$  is conjugated (via  $F_E$ ) to the restriction of  $(\phi_{\mathcal{H}}^t)$  to the zero level  $\{\mathcal{H} = 0\}$ .

Similarly to what we did in the previous section for the construction of  $\mathcal{G}$ , we now extend  $\mathcal{L}$  to a Lagrangian submanifold  $\mathcal{Y}$  of  $T^*\mathcal{M}_1$  invariant by the flow  $(\phi_{\mathcal{H}}^t)$ . The only change is that we choose the lift in such a way that  $\mathcal{Y} \subset \{\mathcal{H} = c\}$  for the constant  $c$  that we introduced in Proposition 5 and Corollary 3.

$$\mathcal{Y} = \{\phi_{\mathcal{H}}^t(q, p, 0, -H(q, p, 0) + c); (q, p) \in \mathcal{L}, t \in [0, 1]\}.$$

Since  $\mathcal{L}$  is invariant by  $\phi_H^{0,1}$ ,  $\mathcal{Y}$  is a closed submanifold of  $T^*\mathcal{M}_1$ . Observe that  $\mathcal{Y}$  is contained in the energy level  $\{\mathcal{H} = c\}$ .

**Proposition 7** *The manifold  $\mathcal{Y}$  is exact Lagrangian, i.e. the Liouville 1-form  $\theta = \langle p, dq \rangle + Ed\tau$  has a primitive  $\mathcal{S}$  along  $\mathcal{Y}$ .*

*Proof* Let  $\tilde{\mathcal{Y}}$  be the set  $\tilde{\mathcal{Y}} = \{(q, p, t, e) \in T^*M \times [0, 1] \times \mathbb{R}; (q, p, [t]_1, e) \in \mathcal{Y}\}$ .

We define the map  $\psi : T^*M \times [0, 1] \times \mathbb{R} \rightarrow T^*\mathcal{M}_2$  by

$$\psi(q, p, t, e) = (q, p, \eta^{-1}(t), \dot{\eta}(\eta^{-1}(t))(e - c)). \quad (14)$$

**Lemma 3**  $\psi|_{\tilde{\mathcal{Y}}}$  is an homeomorphism from  $\tilde{\mathcal{Y}}$  onto  $\mathcal{G} \cap (T^*M \times [0, 1] \times \mathbb{R})$ .

*Proof* Let  $(q, p, t, e) \in \tilde{\mathcal{Y}}$ . This means that  $e = c - H(q, p, t)$  and  $\phi_H^{t,0}(q, p) \in \mathcal{L}$ . Then  $\psi(q, p, t, c - H(q, p, t)) = (q, p, \eta^{-1}(t), -K(q, p, \eta^{-1}(t)))$  with  $\phi_K^{\eta^{-1}(t),0}(q, p) \in \mathcal{L}$ . Hence  $\psi(\tilde{\mathcal{Y}})$  is  $\mathcal{G} \cap (TM \times [0, 1] \times \mathbb{R})$ .

The continuity and injectivity are straightforward.

We define then  $s_0$  by

$$s_0(q, p, t, e) = S \circ i_S^{-1} \circ \psi(q, p, t, e) + ct.$$

Because of equality (6), we have

$$s_0(q, p, 1, e) = S \circ i_S^{-1}(q, p, 1, 0) + c = S \circ i_S^{-1}(q, p, 0, 0) = s_0(q, p, 0, e).$$

Hence we can define  $\mathcal{S} : \mathcal{Y} \rightarrow \mathbb{R}$  by  $\mathcal{S}(q, p, [t]_1, e) = s_0(q, p, t, e)$ .

This function  $\mathcal{S}$  is continuous on  $\mathcal{Y}$  and is differentiable except on the slice  $\mathcal{Y} \cap \{t = 0\}$ . We have  $d\mathcal{S}(q, p, t, e)(\delta q, \delta p, \delta t, \delta e) =$

$$d(S \circ i_S^{-1})(q, p, \eta^{-1}(t), \dot{\eta}(\eta^{-1}(t))(e - c))(\delta q, \delta p, \frac{1}{\dot{\eta}(\eta^{-1}(t))} \delta t, \delta E) + c\delta t,$$

with  $\delta E = \dot{\eta}(\eta^{-1}(t))\delta e + (e - c)\frac{\ddot{\eta}(\eta^{-1}(t))}{\dot{\eta}(\eta^{-1}(t))}\delta t$ .

As  $S \circ i_S^{-1}$  is a primitive of the Liouville 1-form  $\Theta = \langle p, dq \rangle + Ed\tau$ , we deduce that

$$d\mathcal{S}(q, p, t, e)(\delta q, \delta p, \delta t, \delta e) = \langle p, \delta q \rangle + \dot{\eta}(\eta^{-1}(t))(e - c)\frac{\delta t}{\dot{\eta}(\eta^{-1}(t))} + c\delta t = \langle p, \delta q \rangle + e\delta t.$$

Hence  $\mathcal{S}$  is continuous on  $\mathcal{Y}$  and is a primitive of  $\theta$  on  $\mathcal{Y} \setminus \{t = 0\}$ .

As  $\mathcal{Y}$  is Lagrangian, a primitive of  $\theta$  along  $\mathcal{Y}$  exists always locally and is  $C^1$ . Then for every point in  $\mathcal{Y}$ , there exists a connected open neighborhood  $\mathcal{V}$  on which  $\theta$  has a  $C^1$  primitive  $s$ . Without loss of generality we can assume that  $\mathcal{V} \setminus \{t = 0\}$

is made by one or two (open) connected components  $\mathcal{V}_1, \mathcal{V}_2$  (that may be equal). Observe that  $\overline{\mathcal{V}_1} \cap \overline{\mathcal{V}_2} \neq \emptyset$ . On each of these open and connected components  $\mathcal{V}_i$ ,  $\mathcal{S} - s$  is differentiable with its differential equal to 0, hence  $(\mathcal{S} - s)|_{\mathcal{V}_i}$  is equal to a constant  $c_i$ . As  $\mathcal{S} - s$  is continuous, we have also  $(\mathcal{S} - s)|_{\overline{\mathcal{V}_i}} = c_i$ . As  $\overline{\mathcal{V}_1} \cap \overline{\mathcal{V}_2} \neq \emptyset$ , we have  $c_1 = c_2$  and then  $\mathcal{S} - s$  is constant on  $\mathcal{V}$ , therefore  $\mathcal{S}$  is  $C^1$  everywhere and is a primitive of the Liouville 1-form  $\theta$ .

As the exact Lagrangian  $\mathcal{G}$  has a graph selector, the same is true for  $\mathcal{Y}$ .

**Proposition 8** *The function  $u$  is differentiable at every  $z = (q, t) \in \mathcal{U}_0$ , where  $\mathcal{U}_0$  is the open subset defined in (9) and moreover*

$$\forall z \in \mathcal{U}_0, \quad J_u(z) \in \mathcal{Y} \quad \text{and} \quad u(z) = \mathcal{S}(J_u(z)), \quad (15)$$

where  $J_u(z) = (q, d_q u(z), t, \frac{\partial u}{\partial t}(z)) \in T^*M \times \mathbb{T} \times \mathbb{R} = T^* \mathcal{M}_1$ . Moreover identity (15) holds for every  $z \in \mathcal{M}_1$  where  $u$  is differentiable and where  $\mathcal{H}(J_u(z)) = c$ .

*Proof* *Proof that  $u$  is a graph selector.*

NOTATION. Let  $h : \mathcal{M}_1 \rightarrow \mathcal{M}_1$  be defined by  $h(q, t) = (q, \eta^{-1}(t))$ .

Observe that  $h$  is an homeomorphism and that  $h|_{M \times (0,1)}$  is a diffeomorphism onto  $M \times (0, 1)$ . By definition, the function  $u$  is differentiable on every  $z = (q, t) \in \mathcal{U}_0$  and moreover

$$J_u(z) = (q, d_q u(h(z)), t, c + \frac{1}{\eta(\eta^{-1}(t))} \frac{\partial u}{\partial t}(h(z))).$$

As  $u$  is a graph selector for  $\mathcal{G}$ , we have

$$\psi(J_u(z)) = (q, d_q u(h(z)), \eta^{-1}(t), \frac{\partial u}{\partial t}(h(z))) = J_u(h(z)) \in \mathcal{G} \cap (T^*M \times (0, 1) \times \mathbb{R}).$$

By construction of  $\psi$  and by Lemma 3, we can say that  $\psi$  maps  $\mathcal{Y} \cap (T^*M \times (0, 1) \times \mathbb{R})$  diffeomorphically onto  $\mathcal{G} \cap (T^*M \times (0, 1) \times \mathbb{R})$ , therefore  $J_u(z) \in \mathcal{Y}$ .

Moreover we have

$$u(z) = u(h(z)) + ct = S \circ i_S^{-1}(J_u(h(z))) + ct = S \circ i_S^{-1} \circ \psi(J_u(z)) + ct = \mathcal{S}(J_u(z)).$$

*Proof that Identity (15) holds for every  $z \in \mathcal{M}_1$  where  $u$  is differentiable and where  $\mathcal{H}(J_u(z)) = c$ .*

Let  $z = (q, t) \in \mathcal{M}_1$  be a point where  $u$  is differentiable and  $\mathcal{H}(J_u(z)) = c$ . We follow the same step as in [1], and we introduce two subsets of  $T_z^* \mathcal{M}_1 = T_q^* M \times \mathbb{R}$ . Let  $K_u(z)$  be the set of all limit points of sequences  $(du(z_n))_{n \in \mathbb{N}}$  where  $z_n \in \mathcal{U}_0$  and  $\lim_{n \rightarrow +\infty} z_n = z$ , and let  $C_u(z)$  be the convex hull of  $K_u(z)$ . Let us give a result due to F. Clarke (see [12] for a proof of [10] for a more general result).

**Lemma 4** *Let  $f : U \rightarrow \mathbb{R}$  be a Lipschitz function defined on a open subset  $U$  of  $\mathbb{R}^d$  and let  $U_0 \subset U$  be a subset with full Lebesgue measure such that  $f$  is differentiable at every point of  $U_0$ . We introduce a notation. If  $q \in U$ ,  $K_f(q)$  is the set of all the limits  $\lim_{n \rightarrow \infty} df(q_n)$  where  $q_n \in U_0$ ,  $\lim_{n \rightarrow \infty} q_n = q$  and  $C_f(q)$  is the convex hull of  $K_f(q)$ . Then, at every point  $q \in U$  where  $f$  is differentiable, we have :  $df(q) \in C_f(q)$ .*

By hypothesis the function  $p \mapsto H(q, p, t)$  is strictly convex, therefore the energy sublevel

$$\mathcal{H}_{(q,t)}^{-1}((-\infty, c]) = \left\{ (p, e) \in T_{(q,t)}^*(M \times \mathbb{T}_1), \quad H(q, p, t) + e \leq c \right\}$$

is also strictly convex (up to the symmetry  $e \mapsto -e$ , it is the epigraph of the function  $p \mapsto H(q, p, t) - c$ ), and in particular, every point  $(p, e)$  in the energy level  $\mathcal{H}_{(q,t)}^{-1}(c)$  is extremal for  $\mathcal{H}_{(q,t)}^{-1}((-\infty, c])$ . By hypothesis,  $du(z)$  is in the energy level  $\mathcal{H}_z^{-1}(c)$ , therefore it is an extremal point of  $C_u(z)$ , hence a point of  $K_u(z)$ , and by definition of  $K_u(z)$  there exist a sequence  $(z_n)_{n \in \mathbb{N}}$  of points of  $\mathcal{U}_0$  such that  $(z_n, du(z_n))_{n \in \mathbb{N}}$  converges to  $(z, du(z))$ , or equivalently  $(J_u(z_n))_{n \in \mathbb{N}}$  converges to  $(J_u(z))$ , but every point  $J_u(z_n)$  lies in  $\mathcal{Y}$  and satisfies the identity  $u(z_n) = \mathcal{S}(J_u(z_n))$ . Taking the limit  $n \rightarrow +\infty$  we get (15).

## 4 Calibration and conclusion

In this section, we will prove that  $\mathcal{Y}$  is contained in the graph of  $du$ .

Observe that

- the projection of  $\mathcal{Y}$  is compact because  $\mathcal{Y}$  is compact;
- the projection of  $\mathcal{Y}$  is dense in  $\mathcal{M}_1$ . Indeed, Proposition 8 implies that this projection contains  $\mathcal{U}_0$ , which is dense in  $\mathcal{M}_1$ .

Hence the projection of  $\mathcal{Y}$  is  $\mathcal{M}_1$  and we will conclude that  $\mathcal{Y}$  is a graph above the whole  $\mathcal{M}_1$  and that  $u$  is everywhere differentiable. Thus  $\mathcal{Y}$  is the the graph of  $du$ .

Moreover, we will also prove that  $\mathcal{Y}$  is a locally Lipschitz graph in  $T^*\mathcal{M}_1$ . Hence  $\mathcal{Y}$  is a  $C^1$  manifold that is the graph of a locally Lipschitz map. As  $\mathcal{M}$  is compact, this implies that  $\mathcal{Y}$  is the graph of a  $C^1$  map, i.e. that  $u$  is  $C^2$  and  $\mathcal{Y}$  is the graph a  $C^1$  exact 1-form:  $du$ .

The main tool that we will use is the notion of calibrated curve.

### 4.1 Calibration

We will explain what happens along the curves that satisfy the equality in the inequality of domination (7). The proof is an analogue of the proof given by A. Fathi in [11] in the autonomous case.

**DEFINITION.** If  $\gamma : [a, b] \rightarrow M$  is a  $C^1$  arc, its *defect of calibration* is

$$\delta(\gamma) = \int_a^b (L(\gamma(t), \dot{\gamma}(t), t) + c) - (u(\gamma(b), b) - u(\gamma(a), a)).$$

Then

- $\delta$  is always non-negative;
- if  $(\gamma_n)$   $C^1$ -converges to  $\gamma$ , then  $\lim_{n \rightarrow \infty} \delta(\gamma_n) = \delta(\gamma)$ ;
- if  $I \subset J$ , then  $\delta(\gamma_I) \leq \delta(\gamma_J)$ .

DEFINITION. A  $C^1$  curve  $\gamma : I \rightarrow M$  is  $(u, L, c)$ -calibrated if  $\forall [a, b] \subset I$ ,  $\delta(\gamma_{[a,b]}) = 0$ .

**Proposition 9** *If  $\gamma : I \rightarrow M$  is  $(u, L, c)$ -calibrated, then*

- $u$  is differentiable at every  $(\gamma(t), t)$  with  $t$  in the interior of  $I$ ;
- for all  $t$  in the interior of  $I$ , we have  $d_q u(\gamma(t), t) = \frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t)$  and  $\mathcal{H}(J_u(\gamma(t), t)) = c$ .

*Proof* We assume that  $\gamma$  is  $C^1$  and calibrated.

Value of  $du(\gamma(\cdot), \cdot)$  if  $u$  is differentiable along  $t \mapsto (\gamma(t), t)$ .

Let us assume that  $u$  is differentiable at every point of  $\{(\gamma(t), t); t \in (a, b)\}$ .

We have

$$\forall t \in (a, b), u(\gamma(t), t) - u(\gamma(a), a) = \int_a^t (L(\gamma(s), \dot{\gamma}(s), s) + c) ds.$$

Differentiating with respect to  $t \in (a, b)$ , we obtain

$$d_q u(\gamma(t), t) \cdot \dot{\gamma}(t) + \frac{\partial u}{\partial t}(\gamma(t), t) = L(\gamma(t), \dot{\gamma}(t), t) + c.$$

Using Young inequality, we deduce

$$\begin{aligned} c &= d_q u(\gamma(t), t) \cdot \dot{\gamma}(t) + \frac{\partial u}{\partial t}(\gamma(t), t) - L(\gamma(t), \dot{\gamma}(t), t) \\ &\leq \frac{\partial u}{\partial t}(\gamma(t), t) + H(\gamma(t), d_q u(\gamma(t), t), t) = \mathcal{H}(\gamma(t), t, du(\gamma(t), t)). \end{aligned} \quad (16)$$

But Lemma 4 implies that  $\mathcal{H}(\gamma(t), t, du(\gamma(t), t)) \leq c$ . Hence Inequality (16) is in fact an equality. In particular we have equality in Young inequality

$$d_q u(\gamma(t), t) \cdot \dot{\gamma}(t) = L(\gamma(t), \dot{\gamma}(t), t) + H(\gamma(t), d_q u(\gamma(t), t), t)$$

then  $d_q u(\gamma(t), t) = \frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t)$  and so  $\frac{\partial u}{\partial t}(\gamma(t), t) = c - H(\gamma(t), d_q u(\gamma(t), t), t)$ . This can be written  $\mathcal{H}(J_u(\gamma(t), t)) = c$ .

*Proof that  $u$  is differentiable at every  $(\gamma(t), t)$  with  $t$  in the interior of  $I$*

Let us fix  $t_0 \in (a, b)$ . We work in a chart around  $\gamma(t_0) = x$ . Then for every  $t \in (a, b)$  and  $y$  close to  $x$ , we consider the arc  $\gamma_{y,t} : [a, t] \rightarrow M$  that is defined by  $\gamma_{y,t}(s) = \gamma(s) + \frac{s-a}{t-a}(y - \gamma(t))$ . The domination property implies that

$$u(y, t) \leq \psi_+(y, t) = u(\gamma(a), a) + \int_a^t (L(\gamma_{y,t}(s), \dot{\gamma}_{y,t}(s), s) + c) ds.$$



Observe that  $u(x, t_0) = u(\gamma(a), a) + \int_a^{t_0} (L(\gamma(s), \dot{\gamma}(s), s) + c) ds = \psi_+(x, t_0)$  because  $\gamma$  is calibrated. Observe to that

$$\psi_+(y, t) = u(\gamma(a), a) + \int_a^t (L(\gamma(s) + \frac{s-a}{t-a}(y-\gamma(t)), \dot{\gamma}(s) + \frac{1}{t-a}(y-\gamma(t)), s) + c) ds$$

and thus  $\psi_+$  is  $C^1$ .

Let us now consider the arc  $\eta_{y,t} : [t, b] \rightarrow M$  that is defined by  $\eta_{y,t}(s) = \gamma(s) + \frac{b-s}{b-t}(y-\gamma(t))$ . Then

$$\psi_-(y, t) = u(\gamma(b), b) - \int_t^b (L(\eta_{y,t}(s), \dot{\eta}_{y,t}(s), s) + c) ds \leq u(y, t).$$

$\psi_-$  is  $C^1$  and because  $\gamma$  is calibrated we have  $\psi_-(x, t_0) = u(x, t_0)$ .

Finally, we have found two  $C^1$  function  $\psi_-$  and  $\psi_+$  such that  $\psi_- \leq u \leq \psi_+$  and  $\psi_-(x, t_0) = u(x, t_0) = \psi_+(x, t_0)$ . This implies that  $u$  is differentiable at  $(x, t_0)$ .

#### 4.2 Study along the $\Omega$ -limit set of $\phi_{H|\mathcal{L}}^{0,1}$

**Proposition 10** *Let  $(q, p) \in \Omega(\phi_{H|\mathcal{L}}^{0,1})$  be a point of the  $\Omega$ -limit set of  $\phi_{H|\mathcal{L}}^{0,1}$  and let  $(q, p, 0, c - H(q, p, 0))$  be the corresponding point in  $\mathcal{Y}$ . Then the projection of the  $\mathcal{H}$  orbit of  $(q, p, 0, c - H(q, p, 0))$  on  $M$  is  $(u, c, L)$ -calibrated.*

*Proof* Let us fix  $[a, b] \subset \mathbb{R}$  and let us consider the piece of orbit

$$t \in [a, b] \mapsto \zeta(t) = \phi_{\mathcal{H}}^t(q, p, 0, c - H(q, p, 0)) = (q(t), p(t), t, c - H(q(t), p(t), t)).$$

Because  $(q, p) \in \Omega(\phi_{H|\mathcal{L}}^{0,1})$ , we can find a sequence of pieces of  $\mathcal{H}$  orbits

$$t \in [a, b_n] \mapsto \zeta_n(t) = (q_n(t), p_n(t), t, c - H(q_n(t), p_n(t), t))$$

in  $\mathcal{Y}$  such that  $b_n - a \in \mathbb{N}$ ,  $\lim_{n \rightarrow \infty} b_n = +\infty$ ,  $\lim_{n \rightarrow \infty} (q_n(a), p_n(a)) = (q(a), p(a))$  and  $\lim_{n \rightarrow \infty} (q_n(b_n), p_n(b_n)) = (q(a), p(a))$ .

Because of the properties of the defect of calibration  $\delta$ , we have

$$0 \leq \delta(q_{|[a,b]}) = \lim_{n \rightarrow \infty} \delta(q_n|_{[a,b]}) \leq \liminf_{n \rightarrow \infty} \delta(q_n|_{[a,b_n]}).$$

We have

$$\delta(q_n|_{[a,b_n]}) = u(q_n(b_n), b_n) - u(q_n(a), a) - \int_a^{b_n} (L(q_n(t), \dot{q}_n(t), t) + c) dt.$$

We prove now the following lemma.

**Lemma 5** *If  $t \in \mathbb{R} \mapsto \zeta(t) = (q(t), p(t), t, c - H(q(t), p(t), t))$  is an orbit for  $\mathcal{H}$  on  $\mathcal{Y}$ , then we have*

$$\int_a^b (L(q(t), \dot{q}(t), t) + c) = \mathcal{S}(\zeta(b)) - \mathcal{S}(\zeta(a)). \quad (17)$$

Because of Proposition 7, we know that  $\mathcal{S}$  is a primitive of  $\theta$  along  $\mathcal{Y}$  and so we have

$$\mathcal{S}(\zeta(b)) - \mathcal{S}(\zeta(a)) = \int_{\zeta|_{[a,b]}} \theta = \int_a^b (p(t) \cdot \dot{q}(t) - H(q(t), p(t), t) + c) dt$$

and along every orbit we have  $p(t) \cdot \dot{q}(t) - H(q(t), p(t), t) = L(q(t), \dot{q}(t), t)$ . This proves the lemma.

Applying Lemma 5, we obtain:

$$\delta(q_{n|[a,b_n]}) = u(q_n(b_n), b_n) - u(q_n(a), a) - (\mathcal{S}(\zeta_n(b_n)) - \mathcal{S}(\zeta_n(a))).$$

Using the continuity of  $u$  and  $\mathcal{S}$  and the fact that  $\lim_{n \rightarrow \infty} \zeta_n(a) = \zeta(a)$  and  $\lim_{n \rightarrow \infty} \zeta_n(b_n) = \zeta(b)$ , we deduce that

$$\lim_{n \rightarrow \infty} \delta(q_{n|[a,b_n]}) = u(q(a), a) - u(q(a), a) - (\mathcal{S}(\zeta(a)) - \mathcal{S}(\zeta(a))) = 0$$

and thus  $q$  is calibrated.

#### 4.3 Every orbit in $\mathcal{Y}$ is in the graph of $du$

**Proposition 11** *Let  $t \mapsto \zeta(t) = \phi_{\mathcal{H}}^t(q, p, 0, c - H(q, p, 0)) = (q(t), p(t), t, c - H(q(t), p(t), t))$  be an orbit for  $\mathcal{H}$  on  $\mathcal{Y}$ . Then the curve  $q(t)$  is  $(u, c, L)$ -calibrated and we have*

$$\forall t \in \mathbb{R}, (p(t), c - H(q(t), p(t), t)) = du(q(t), t).$$

*Proof* We choose  $(q_+, p_+) \in \omega((q(0), p(0)), \Phi_H^{0,1})$  and  $(q_-, p_-) \in \alpha((q(0), p(0)), \Phi_H^{0,1})$  and we denote by  $\zeta_{\pm}(t) = (q_{\pm}(t), p_{\pm}(t), t, c - H(q_{\pm}(t), p_{\pm}(t), t))$  the corresponding  $\mathcal{H}$  orbits in  $\mathcal{Y}$ .

Then there exists two increasing sequences  $(n_i)$  and  $(m_i)$  of positive integers so that

$$\lim_{i \rightarrow \infty} \zeta(-m_i) = \zeta_-(0) \quad \text{and} \quad \lim_{i \rightarrow \infty} \zeta(n_i) = \zeta_+(0). \quad (18)$$

If  $[a, b] \subset \mathbb{R}$ , we have

$$0 \leq \delta(q_{|[a,b]}) \leq \liminf_{i \rightarrow \infty} \delta(q_{[-m_i, n_i]}),$$

hence we will prove that this last limit is zero.

Using Lemma 5, we obtain

$$\begin{aligned} \delta(q_{[-m_i, n_i]}) &= \int_{-m_i}^{n_i} (L(q(t), \dot{q}(t), t) + c) dt - (u(q(n_i), n_i) - u(q(-m_i), -m_i)) \\ &= \mathcal{S}(\zeta(n_i)) - u(q(n_i), n_i) - (\mathcal{S}(\zeta(-m_i)) - u(q(-m_i), -m_i)) \end{aligned}$$

Because of (18) and of the continuity of  $u$  and  $\mathcal{S}$ , we obtain

$$\lim_{i \rightarrow \infty} \delta(q_{[-m_i, n_i]}) = \mathcal{S}(\zeta_+(0)) - u(q_+(0), 0) - (\mathcal{S}(\zeta_-(0)) - u(q_-(0), 0)).$$

We deduce from Proposition 10 and Proposition 9 that  $u$  is differentiable at every  $(q_{\pm}(t), t)$  and that

$$d_q u(q_{\pm}(t), t) = \frac{\partial L}{\partial v}(q_{\pm}(t), \dot{q}_{\pm}(t), t) \quad \text{and} \quad \mathcal{H}(q_{\pm}(t), d_q u(q_{\pm}(t), t), t, \frac{\partial u}{\partial t}(q_{\pm}(t), t)) = c.$$

This implies that  $du(q_{\pm}(t), t) = (p_{\pm}(t), c - H(q_{\pm}(t), p_{\pm}(t), t))$  and then that  $\zeta_{\pm}(t) = (q_{\pm}(t), d_q u(q_{\pm}(t), t), t, \frac{\partial u}{\partial t}(q_{\pm}(t), t))$ . This gives that

$$\begin{aligned} \lim_{i \rightarrow \infty} \delta(q_{[-m_i, n_i]}) &= \mathcal{S}(q_+(0), d_q u(q_+(0), 0), 0, \frac{\partial u}{\partial t}(q_+(0), 0)) - u(q_+(0), 0) \\ &\quad - (\mathcal{S}(q_-(0), d_q u(q_-(0), 0), 0, \frac{\partial u}{\partial t}(q_-(0), 0)) - u(q_-(0), 0)). \end{aligned}$$

Proposition 8 tells us that

$$\forall t \in \mathbb{R}, (q_{\pm}(t), d_q u(q_{\pm}(t), t), t, \frac{\partial u}{\partial t}(q_{\pm}(t), t)) \in \mathcal{Y}$$

and

$$u(q_{\pm}(t), t) = \mathcal{S}(q_{\pm}(t), d_q u(q_{\pm}(t), t), t, \frac{\partial u}{\partial t}(q_{\pm}(t), t)).$$

We finally deduce that

$$\lim_{i \rightarrow \infty} \delta(q_{[-m_i, n_i]}) = 0$$

and that  $q$  is  $(u, L, c)$ -calibrated.

We deduce from Proposition 9 that  $u$  is differentiable at every  $(q(t), t)$  and that

$$d_q u(q(t), t) = \frac{\partial L}{\partial v}(q(t), \dot{q}(t), t) \quad \text{and} \quad \mathcal{H}(q(t), d_q u(q(t), t), t, \frac{\partial u}{\partial t}(q(t), t)) = c.$$

This implies that  $\zeta(t) = (q(t), d_q u(q(t), t), t, \frac{\partial u}{\partial t}(q(t), t)) = J_u(q(t), t)$ .

#### 4.4 The Lipschitz property of $\mathcal{Y}$

We know that  $\mathcal{Y}$  is the graph  $du$ . We wish to show that  $du$  is locally Lipschitz. Then we will deduce that  $\mathcal{L}$ , which is a  $C^1$  Lagrangian submanifold and a locally Lipschitz graph, is the graph of a  $C^1$  1-form.

We use Proposition 4.11.3 of [11]:

**Criterion for a Lipschitz derivative (Albert Fathi)**. *Let  $B = B(x_0, r_0)$  be an open ball in  $\mathbb{R}^n$ , let  $u : B \rightarrow \mathbb{R}$  be a function and let  $K > 0$  be a positive constant. We introduce the following notation*

$$A_{K,u} = \{x \in B; \exists \varphi_x \in L(\mathbb{R}^n, \mathbb{R}), \forall y \in B, |u(y) - u(x) - \varphi_x(y-x)| \leq K\|y-x\|^2\}.$$

*Then  $u$  has a derivative at every point of  $A_{K,u}$  and if  $x \in A_{K,u}$ , we have  $d_x u = \varphi_x$ . Moreover, the restriction of  $x \mapsto d_x u$  to  $\{x \in A_{K,u}; \|x - x_0\| \leq \frac{r_0}{3}\}$  is Lipschitz with a*

Lipschitz constant equal to  $6K$ .

Let us fix  $(x_0, t_0) \in M \times [0, 1]$  and let us prove that  $du$  is Lipschitz in some neighbourhood of  $(x_0, t_0)$ . We work in some chart, i.e.  $B_0 = \bar{B}(x_0, r_0)$ . Then we choose  $B_1 = \bar{B}(x_0, r_1) \subsetneq B_0$  and  $I_0 = [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ , such that

$$\forall t, \tau \in I_0, \forall x \in B_1, \pi \circ \varphi_H^{t, \tau}(x, d_q u(x, t)) \in B_0.$$

Let us set  $\varepsilon_1 = \frac{\varepsilon_0}{4}$  and  $I_1 = [t_0 - \varepsilon_1, t_0 + \varepsilon_1]$ . If  $(x, t) \in B_1 \times I_1$ , we consider the  $(u, L, c)$ -calibrated curve

$$\Gamma = \Gamma_{x,t} : [t - 3\varepsilon_1, t + 3\varepsilon_1] \rightarrow B_0 \\ s \mapsto \pi \circ \varphi_H^{t,s}(x, d_q u)(x, t).$$

Then, for every  $(y, \tau) \in B_1 \times I_1$ , we can build two curves (as we did in the proof of Proposition 9)

–  $\gamma_{y,\tau} = \gamma_{y,\tau,x,t} : [t - 3\varepsilon_1, \tau] \rightarrow B_0$  defined by

$$\gamma_{y,\tau}(s) = \Gamma_{x,t}(s) + \frac{s - (t - 3\varepsilon_1)}{\tau - (t - 3\varepsilon_1)}(y - \Gamma_{x,t}(\tau));$$

–  $\eta_{y,\tau} = \eta_{y,\tau,x,t} : [\tau, t + 3\varepsilon_1] \rightarrow B_1$  defined by

$$\eta_{y,\tau}(s) = \Gamma_{x,t}(s) + \frac{(t + 3\varepsilon_1) - s}{(t + 3\varepsilon_1) - \tau}(y - \Gamma_{x,t}(\tau)).$$

Observe that  $\gamma_{x,t} = \Gamma_{x,t} \big|_{[t-3\varepsilon_1, \tau]}$  and  $\eta_{x,t} = \Gamma_{x,t} \big|_{[\tau, t+3\varepsilon_1]}$ . We then define

$$\psi_+(y, \tau) = \psi_{+,x,t}(y, \tau) = u(\Gamma_{x,t}(t - 3\varepsilon_1), t - 3\varepsilon_1) + \int_{t-3\varepsilon_1}^{\tau} (L(\gamma_{y,\tau}(s), \dot{\gamma}_{y,\tau}(s), s) + c) ds$$

and

$$\psi_-(y, \tau) = \psi_{-,x,t}(y, \tau) = u(\Gamma_{x,t}(t + 3\varepsilon_1), t + 3\varepsilon_1) - \int_{\tau}^{t+3\varepsilon_1} (L(\eta_{y,\tau}(s), \dot{\eta}_{y,\tau}(s), s) + c) ds.$$

Let us recall that  $\psi_- \leq u \leq \psi_+$ , that  $\psi_-$  and  $\psi_+$  are  $C^1$  and that  $\psi_-(x, t) = u(x, t) = \psi_+(x, t)$ . In particular,

$$u(y, \tau) - u(x, t) \leq \psi_+(y, \tau) - \psi_+(x, t) = \alpha + c(\tau - t) + \beta \quad (19)$$

where

$$\alpha = \int_{t-3\varepsilon_1}^{\tau} (L(\gamma_{y,\tau}(s), \dot{\gamma}_{y,\tau}(s), s) - L(\gamma_{x,t}(s), \dot{\gamma}_{x,t}(s), s)) ds$$

and

$$\beta = \int_{\tau}^{t+3\varepsilon_1} L(\eta_{y,\tau}(s), \dot{\eta}_{y,\tau}(s), s) ds.$$

Because each curves  $s \mapsto \frac{\partial L}{\partial v}(\Gamma_{x,t}(s), \dot{\Gamma}_{x,t}(s), s)$  is drawn on  $\mathcal{U}$  and then bounded, there exists a constant  $K_0$  such that

$$\forall (x, t) \in B_1 \times I_1, \forall s \in [t - 3\varepsilon_1, t + 3\varepsilon_1], \|\dot{\Gamma}_{x,t}(s)\| \leq K_0.$$

Because  $L$  is  $C^2$ , there also exists a constant  $K_1 \geq K_0$  such the first and second derivative of  $L$  are bounded by  $K_1$  on the set

$$B_1 \times B(0, K_0 + \frac{2r_0}{\varepsilon_1}) \times I_1.$$

REMARK. Because  $\dot{\gamma}_{y,\tau}(s) = \dot{\gamma}_{x,t}(s) + \frac{y - \Gamma_{x,t}(\tau)}{\tau - (t - 3\varepsilon_1)}$ , we have

$$\|\dot{\gamma}_{y,\tau}(s)\| \leq K_0 + \frac{1}{\varepsilon_1} \|y - \Gamma_{x,t}(\tau)\| \leq K_0 + \frac{2r_0}{\varepsilon_1} = K_2. \quad (20)$$

In the following of the proof we denote by  $K_3, K_4, K_5 \dots$  some positive constants depending only on the restriction of  $L$  to the set  $B_0 \times B(0, K_2) \times I_0$ . Taylor-Lagrange inequality implies that for every  $s \in [t - 3\varepsilon_1, \tau]$ , we have

$$\begin{aligned} & |L(\gamma_{y,\tau}(s), \dot{\gamma}_{y,\tau}(s), s) - L(\gamma_{x,t}(s), \dot{\gamma}_{x,t}(s), s) - \frac{\partial L}{\partial x}(\gamma_{x,t}(s), \dot{\gamma}_{x,t}(s), s) \frac{s - (t - 3\varepsilon_1)}{\tau - (t - 3\varepsilon_1)} (y - \gamma_{x,t}(\tau)) \\ & - \frac{\partial L}{\partial v}(\gamma_{x,t}(s), \dot{\gamma}_{x,t}(s), s) \frac{y - \gamma_{x,t}(\tau)}{\tau - (t - 3\varepsilon_1)}| \leq \frac{K_3}{\varepsilon_1^2} \|y - \gamma_{x,t}(\tau)\|^2. \end{aligned} \quad (21)$$

Using Euler-Lagrange equations, inequality (21) and an integration by parts, since  $\gamma_{x,t} = \Gamma_{x,t} \big|_{[t-3\varepsilon_1, \tau]}$ , we get the following inequality

$$\left| \alpha - \left[ \frac{\partial L}{\partial v}(\Gamma_{x,t}(s), \dot{\Gamma}_{x,t}(s), s) \frac{s - (t - 3\varepsilon_1)}{\tau - (t - 3\varepsilon_1)} (y - \Gamma_{x,t}(\tau)) \right]_{s=t-3\varepsilon_1}^{s=\tau} \right| \leq \frac{3K_3}{\varepsilon_1} \|y - \Gamma_{x,t}(\tau)\|^2$$

i.e.

$$\left| \alpha - \frac{3\varepsilon_1}{\tau - (t - 3\varepsilon_1)} \frac{\partial L}{\partial v}(\Gamma_{x,t}(t), \dot{\Gamma}_{x,t}(t), t) (y - \Gamma_{x,t}(\tau)) \right| \leq \frac{3K_3}{\varepsilon_1} \|y - \Gamma_{x,t}(\tau)\|^2.$$

We deduce from inequality (20) that  $\|\Gamma_{x,t}(\tau) - \Gamma_{x,t}(t)\| \leq K_2|t - \tau|$  and then  $\|y - \Gamma_{x,t}(\tau)\| = \|(y - x) + (x - \Gamma_{x,t}(\tau))\| \leq \|y - x\| + K_2|t - \tau|$ . We note too that  $|\frac{3\varepsilon_1}{\tau - (t - 3\varepsilon_1)} - 1| \leq \frac{|t - \tau|}{\varepsilon_1}$  and so

$$|\alpha - d_q u(x, t) (y - \Gamma_{x,t}(\tau))| \leq K_4 (\|y - x\| + |t - \tau|)^2.$$

Observe that Euler-Lagrange Equation implies that the  $\dot{\Gamma}_{x,t}$ s are uniformly bounded by some constant. Hence

$$\|y - \Gamma_{x,t}(\tau) - (y - x - \dot{\Gamma}_{x,t}(t)(\tau - t))\| \leq K_5(\tau - t)^2.$$

We deduce

$$|\alpha - d_q u(x, t) (y - x - \dot{\Gamma}_{x,t}(t)(\tau - t))| \leq K_6 (\|y - x\| + |t - \tau|)^2. \quad (22)$$

In a similar way, we obtain

$$|\beta - (\tau - t)L(x, \dot{\Gamma}_{x,t}(t), t)| \leq K_7 (\|y - x\| + |t - \tau|)^2. \quad (23)$$

Equations (19), (22) and (23) imply that

$$u(y, \tau) - u(x, t) - d_q u(x, t)(y - x) - (\tau - t) \frac{\partial u}{\partial t}(x, t) \leq K_8 (\|y - x\| + |t - \tau|)^2.$$

Using  $\psi_-$  instead of  $\psi_+$ , we obtain then

$$|u(y, \tau) - u(x, t) - d_q u(x, t)(y - x) - (\tau - t) \frac{\partial u}{\partial t}(x, t)| \leq K_9 (\|y - x\| + |t - \tau|)^2.$$

Using the criterion for a Lipschitz derivative, we conclude.

#### 4.5 Proof of the corollaries

We prove the corollaries that were given in the introduction.

**Proof or Corollary 1** With the hypothesis of the corollary, we obtain that  $\mathcal{L}$  is a graph. We then use Theorem 6.4.1 of [11], which is a corollary of the convergence of the Lax-Oleinik semi-group in weak KAM theory, to conclude.

**Proof or Corollary 2** We proved that the  $\mathcal{H}$  orbit of every point in  $\mathcal{Y}$  is  $(u, L, c)$ -calibrated. This implies (see for example [3]) that every orbit is minimizing.

**Acknowledgements.** The authors thank sincerely the referee for many valuable suggestions and comments, who greatly improve the the quality of the paper.

#### References

1. M.-C. Arnaud, *On a Theorem due to Birkhoff*, *Geom. Funct. Anal.* Vol. **20**, 1307-1316 (2010).
2. M.-C. Arnaud, *When are the invariant submanifolds of symplectic dynamics Lagrangian?* *Discrete Contin. Dyn. Syst.* **34**, no. 5, 1811-1827 (2014)
3. P. Bernard, *The dynamics of pseudographs in convex Hamiltonian systems*. *J. Amer. Math. Soc.* 21 (2008), no. 3, 615–669.
4. P. Bernard, J. O. Santos *A geometric definition of the Mañé-Mather set and a Theorem of Marie-Claude Arnaud*. *Math. Proc. Cambridge Philos. Soc.* 152 (2012), no. 1, 167–178.
5. M. Bialy & L. Polterovich. *Hamiltonian diffeomorphisms and Lagrangian distributions*, *Geom. Funct. Anal.* 2, 173–210 (1992)
6. M. Bialy & L. Polterovich. *Hamiltonian systems, Lagrangian tori and Birkhoff's theorem*. *Math. Ann.* 292 (1992), no. 4, 619–627.
7. G. D. Birkhoff. *Surface transformations and their dynamical application*. *Acta Math.* **43** (1920) 1-119.
8. M. Brunella, *On a theorem of Sikorav*, *Enseign. Math.* (2) **37** (1991), no. 1-2, 83-87
9. M. Chaperon, *Lois de conservation et géométrie symplectique*, *C.R. Acad. Sci.* **312** (1991), 345-348.
10. F. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, 1983.
11. A. Fathi, *Weak KAM theorems in Lagrangian dynamics*, book in preparation.
12. A. Fathi, & E. Maderna, *Weak KAM theorem on non compact manifolds*, *NoDEA Nonlinear Differential Equations Appl.* **14** (2007), no. 1-2, 1-27.
13. C. Golé, *Symplectic twist maps. Global variational techniques*. *Advanced Series in Nonlinear Dynamics*, 18. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xviii+305 pp
14. M. Herman, *Sur les courbes invariantes par les difféomorphismes de l'anneau*, Vol. 1, *Asterisque* **103-104** (1983).

15. M. Herman, *Inégalités "a priori" pour des tores lagrangiens invariants par des difféomorphismes symplectiques*, vol. I, Inst. Hautes Études Sci. Publ. Math. No. **70**, 47–101 (1989)
16. M. Herman, *Dynamics connected with indefinite normal torsion. Twist mappings and their applications*, 153182, IMA Vol. Math. Appl., **44**, Springer, New York, 1992
17. W. S. Massey, *A basic course in algebraic topology*, Graduate Texts in Mathematics, **127** Springer-Verlag, New York, (1991). xvi+428 pp
18. J. Mather, *Action minimizing measures for positive definite Lagrangian systems*, Math. Z. **207**, 169–207 (1991).
19. J. Moser, *Proof of a generalized form of a fixed point theorem due to G. D. Birkhoff*, Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), pp. 464–494. Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977.
20. G. P. Paternain, L. Polterovich, K. F. Siburg, *Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry-Mather theory*, Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. **3** (2003), no. 2, 593-619
21. K. F. Siburg, *The principle of least action in geometry and dynamics*, Lecture Notes in Mathematics, **1844** (2004) Springer-Verlag, Berlin, xii+128 pp.
22. J.-C. Sikorav, *Problèmes d'intersections et de points fixes en géométrie hamiltonienne*, Comment. Math. Helv. **62** (1987), no. 1, 62-73.
23. D. Théret, *A complete proof of Viterbo's uniqueness theorem on generating functions*, Topology Appl. **96** (1999), no. 3, 249-266.
24. C. Viterbo, *Symplectic topology as the geometry of generating functions*, Math. Ann. **292** (1992), no. 4, 685-710.