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CONFORMALLY COVARIANT PARAMETRIZATIONS FOR INITIAL DATA IN SOME MODIFIED EINSTEIN GRAVITY THEORIES

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Abstract. We propose further conformal parametrizations for initial data in some modified Einstein gravity theories. Some of them give rise to conformally covariant systems.

Keywords : Conformal Riemannian geometry, non-linear elliptic systems, $f(R)$ gravity, Lovelock gravity, Einstein-Gauss-Bonnet gravity, higher order gravity, constraint equations, Cauchy problem.

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1. Introduction

Motivated by the problem of dark matter, early-time inflation or late-time acceleration of the universe, a multitude of modified gravity theories are currently studied: $f(R)$, Lovelock, Einstein-Gauss-Bonnet, cubic, quadratic,...

As in general relativity, the space and time correlation in theses theories implies that the initial data for the evolution problem cannot be chosen freely.

In general, in vacuum for instance, the initial data for such modified Einstein gravity theories are given by a manifold M of dimension n, equipped with a Riemannian metric $\widehat{g}$ (spatial geometry at a time 0) and a symmetric 2-tensor field $\widehat{K}$ (infinitesimal deformation of the spatial geometry at time 0), satisfying some constraint equations of the form (for details, see eg. [4], [1], [7], [8], and the appendix 5.2)

$$\begin{cases} \rho(\widehat{g},\widehat{K}) = 0 \ , \\ J(\widehat{g},\widehat{K}) = 0 \ . \end{cases} \tag{C}$$

In this system, the first equation, the Hamiltonian constraint, is scalar and the second one, the momentum constraint, is vectorial. Such a system is known as the Cauchy problem for the related gravity theory. This system is highly under-determined because it contains $(n+1)$ equations for $n(n+1)$ unknowns. As a consequence, it is natural to

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look for $(n+1)$ unknowns, fixing the remaining ones.

The goal of this note is to point out some different possibilities to parametrize (and then different ways to construct) solutions of (C). Some of them give rise to an interesting mathematical property, namely the conformal covariance.

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2. The usual conformal method

On a smooth manifold M of dimension n, given a Riemannian metric g, we denote by ∇ its Levi-Civita connection. If h is a symmetric covariant 2-tensor field, we define its divergence as the 1-form given by

$$(\mathrm{div}_g\, h)_i = -\nabla^k h_{ki}.$$

The classical method (York's method A) starts from a given metric g together with a trace-free and divergence-free symmetric 2-tensor field σ (a TT-tensor) and a real function[1] τ. It consists in looking for solutions of (C) of the form

$$\widehat{g} = \phi^{N-2} g\ ,\quad \widehat{K} = \frac{\tau}{n}\widehat{g} + \phi^{-2}(\sigma + \mathring{\mathcal{L}}_g W) \tag{P}$$

where $N = \frac{2n}{n-2}$, and the unknowns are a function $\phi > 0$ and a one form W and where

$$(\mathring{\mathcal{L}}_g W)_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n}\nabla^k W_k\, g_{ij}.$$

We infer from (C) and (P) a coupled system of the form (see eg. [1], [8])

$$(S)\begin{cases} \rho(\widehat{g},\widehat{K}) =: L_{g,\tau,\sigma}(\phi, W) = 0 & \text{(L)} \\ J(\widehat{g},\widehat{K}) =: V_{g,\tau,\sigma}(\phi, W) = 0 & \text{(V)} \end{cases}$$

where the scalar equation (L) is a generalisation of the Lichnerowicz one (see [6]) and the equation (V) is usually called the vector equation (see [9]).

In order to produce further interesting parametrization we recall some basic fact. Firstly, the operators div_g and $\mathring{\mathcal{L}}_g$ are conformally covariant (see appendix 5.1 for a precise definition). Indeed, for any positive function ψ, if $\widetilde{g} = \psi^{N-2} g$ then

$$\mathrm{div}_{\widetilde{g}}\, h = \psi^{-N}\, \mathrm{div}_g(\psi^2 h)\ ,\quad \mathring{\mathcal{L}}_{\widetilde{g}} W = \psi^{N-2}\mathring{\mathcal{L}}_g(\psi^{2-N} W). \tag{$\mathcal{C}$}$$

Secondly, we recall the York decomposition [9] valid for instance if M is compact and g has no conformal Killing vector fields (i.e. $\ker\ \mathring{\mathcal{L}}_g$

[1] playing morally the role of a mean curvature function

is trivial): any covariant symmetric trace free 2-tensor field h splits in a unique way as

$$h = \sigma + \mathring{\mathcal{L}}_g W, \tag{Y}$$

where σ is a TT-tensor and W is a 1-form.

3. A conformally covariant parametrization

Getting back to (C), we use now the York decomposition relative to $\widehat{g}$, namely

$$\widehat{K} - \frac{\tau}{n}\widehat{g} = \widehat{\sigma} + \mathring{\mathcal{L}}_{\widehat{g}}\widehat{W}.$$

From $(\mathcal{C})$, if we are looking for a solution of (C) in a conformal class $\widehat{g} = \phi^{N-2}g$, it is natural to introduce $\widehat{\sigma} = \phi^{-2}\sigma$, where σ is a TT-tensor. Still using $(\mathcal{C})$, and expressing $\mathring{\mathcal{L}}_{\widehat{g}}\widehat{W}$ in terms of $g = \phi^{2-N}\widehat{g}$, we are also prompted to set $\widehat{W} := \phi^{N-2}W$.

Sticking to the same fixed g, σ, τ, we can thus parametrize the solutions of the constraint (C) by

$$\begin{cases} \widehat{g} = \phi^{N-2}g \ , \\ \widehat{K} = \dfrac{\tau}{n}\widehat{g} + \phi^{-2}(\sigma + \phi^N \mathring{\mathcal{L}}_g W) \ . \end{cases}$$

With this parametrization in (C) we obtain a new system of the form (see eg. [5])

$$(S'_{g,\tau,\sigma}) \begin{cases} \rho(\widehat{g}, \widehat{K}) =: L'_{g,\tau,\sigma}(\phi, W) = 0 \ , \\ J(\widehat{g}, \widehat{K}) =: V'_{g,\tau,\sigma}(\phi, W) = 0 \ . \end{cases}$$

If we follow the same method but, instead of g, starts from a conformal metric $\widetilde{g} = \psi^{N-2}g$ and from the related symmetric 2-tensor $\widetilde{\sigma} = \psi^{-2}\sigma$ (which is $\mathrm{div}_{\widetilde{g}}$ free by $(\mathcal{C})$), sticking to the same given real function τ, we see from $(\mathcal{C})$ that the couple $(\widetilde{\phi} = \psi^{-1}\phi, \widetilde{W} = \psi^{N-2}W)$ solves the resulting system $(S'_{\widetilde{g},\tau,\widetilde{\sigma}})$ iff (ϕ, W) solves $(S'_{g,\tau,\sigma})$.

In other words,the system $(S'_{g,\tau,\sigma})$ is *conformally covariant*. This simple observation shows that York's method B (see [1, Section 4.1] or the original paper [9, Page 461]), also called the physical TT method, stays valid for every modified gravity theory, regardless of the explicit form of the equations (C).

4. Further parametrizations

In this section, we propose more general but still natural parametrizations. Let $\widetilde{g} = \psi^{N-2}g$ be another conformal metric. The York decomposition (Y) relative to $\widetilde{g}$ of the tensor

$$\phi^2\psi^{-2}(\widehat{K} - \frac{\tau}{n}\widehat{g}),$$

leads to the parametrization

$$\begin{cases} \widehat{g} = \phi^{N-2} g \ , \\ \widehat{K} = \dfrac{\tau}{n}\widehat{g} + \phi^{-2}(\sigma + \psi^N \mathring{\mathcal{L}}_g W). \end{cases}$$

Putting this parametrization in (C), we obtain a new interesting system. Of course, we could let ψ depend on ϕ and possibly on some other parameters in the system obtained. It can be seen (compare [1] for the GR initial data) that the conformal method A consists in choosing $\psi = 1$, the conformal method B arises when $\psi = \phi$, and for ψ a fixed positive function, we obtain the conformal thin sandwich method. But many other choices can be made, and some of them also yield conformally covariants systems (see [5]).

Remarks 4.1.
• It is probable that, depending on the gravity theory studied, an adapted choice of ψ will be judicious.
• For an arbitrary function f, we could use in the same way the York decomposition related to $\widetilde{g}$ of the tensor $f(\widehat{K} - \frac{\tau}{n}\widehat{g})$. This will produce a system where f and ψ can be choosen freely (possibly depending on some other parameters and/or variables).
• In [8], a discussion is made about a choice of the different powers of the conformal factor that can be used to define the conformal parametrization. With our parametrizations, only the $\widehat{A}_{ij}$ there is changed. But we can see that in that case the natural choice to make, with the notations of [8] is $\tau = 0$, $m = 2/(N-2)$ and $l = -2$ there.

5. Appendix

5.1. **Conformal covariance.** Let us consider three products of tensor bundles over M,

$$E = E_1 \times ... \times E_k, \quad F = F_1 \times ... \times F_l, \quad G = G_1 \times ... \times G_m,$$

and a differential operator acting on the sections :

$$P_g : \Gamma(E) \longrightarrow \Gamma(F),$$

with coefficients determined by $g = (g_1, ..., g_m) \in G$. We will say that P_g is conformally covariant if there exist $a = (a_1, .., a_k) \in \mathbb{R}^k$, $b = (b_1, .., b_l) \in \mathbb{R}^l$ and $c = (c_1, .., c_m) \in \mathbb{R}^m$ such that for each smooth section e of E, and every smooth function ψ on M, we have

$$\psi^b \odot P_{\psi^c \odot g}(\psi^a \odot e) = P_g(e),$$

where

$$\psi^a \odot e = (\psi^{a_1} e_1, ..., \psi^{a_k} e_k).$$

A differential system will be said conformally covariant if it can be written in the form $P_g(e) = f$, for a conformally covariant operator P_g.

5.2. **The equations.** We specify here the equations related to the Hamiltonian constraint $\rho(g,K) = 0$ and the momentum constraint $J(g,K) = 0$, in some modified gravity theories.

The constraints equations for the general Lovelock gravity can be found in [3] pages 692-693 (a reedition of [2] pages 56-57). We do not reproduce them here but choose to only details the Einstein-Gauss-Bonnet particular case.

Einstein-Gauss-Bonnet constraint: The following expressions can be found in [8] for instance:

$$\rho(g,K) = M + \alpha_{GB}(M^2 - 4M_{ij}M^{ij} + M_{ijkl}M^{ijkl}),$$

$$\frac{-1}{2}J_i(g,K) = N_i + 2\alpha_{GB}(MN_i - 2M_i^j N_j + 2M^{kl}N_{ikl} - M_i{}^{jkl}N_{klj}),$$

where

$$\begin{aligned} M_{ijkl} &= R_{ijkl}(g) + K_{ik}K_{jl} - K_{il}K_{jk}, \\ M_{ij} &= R_{ij}(g) + \mathrm{Tr}_g K K_{ij} - K_{il}K_j^l, \\ M &= R(g) + (\mathrm{Tr}_g K)^2 - K_{ij}K^{ij}, \\ N_{ijk} &= \nabla_i K_{jk} - \nabla_j K_{ik} \\ N_i &= \nabla_j K_i^j - \nabla_i \mathrm{Tr}_g K, \end{aligned}$$

and α_{GB} is a coupling constant, equal to zero in the Einstein theory.

f(R) gravity constraint: The following equations can be found in [7] :

$$\begin{aligned} \rho(g,K) &= R(g) - K_{ij}K^{ij} + (\mathrm{Tr}_g K)^2 \\ &\quad - f'(\mathcal{R})^{-1}[2\Delta_g f'(\mathcal{R}) + 2\,\mathrm{Tr}_g K f''(\mathcal{R})\dot{\mathcal{R}} - f(\mathcal{R}) - \mathcal{R}f'(\mathcal{R})], \end{aligned}$$

$$\frac{1}{2}J_j(g,K) = -\nabla^i K_{ij} + \nabla_j \mathrm{Tr}_g K - f'(\mathcal{R})^{-1}\nabla_j(f''(\mathcal{R})\dot{\mathcal{R}}) - K_j^i \nabla_i(\ln(f'(\mathcal{R}))),$$

where $\Delta = \nabla^i\nabla_i$, and $\mathcal{R}$, $\dot{\mathcal{R}}$ are respectively the initial data[2] of the scalar curvature and the time derivative of the scalar curvature of the space time. More precisely $\dot{\mathcal{R}}$ will be equal to $\mathcal{L}_n\mathcal{R}$, where n will be the futur unit normal to the initial data of the space-time when evolved.

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[2] to be chosen

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