

# Non-singular spacetimes with a negative cosmological constant: II. Static solutions of the Einstein-Maxwell equations

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# NON-SINGULAR SPACETIMES WITH A NEGATIVE COSMOLOGICAL CONSTANT: II. STATIC SOLUTIONS OF THE EINSTEIN-MAXWELL EQUATIONS

PIOTR T. CHRUSCIEL AND ERWANN DELAY

ABSTRACT. We construct infinite-dimensional families of non-singular static space times, solutions of the vacuum Einstein-Maxwell equations with a negative cosmological constant. The families include an infinite-dimensional family of solutions with the usual AdS conformal structure at conformal infinity.

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## 1. INTRODUCTION

Stationary and static solutions play a fundamental role in any theory. Stable such solutions provide families of possible end states of the problem at hand. Unstable ones display features of the solutions that are likely not to be encountered at late times of evolution.

Lichnerowicz's theorem asserts that the only vacuum stationary well-behaved solution with  $\Lambda = 0$  is Minkowski space-time. It therefore came as a surprise that there exist many stationary solutions of the vacuum Einstein equations with negative cosmological constant which have a smooth

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conformal structure [3, 4, 8]. In particular there exist many globally well behaved stationary vacuum space-times with negative cosmological constant which have no symmetries whatsoever.

Now, Lichnerowicz's theorem generalises to static Einstein-Maxwell equations with  $\Lambda = 0$  [14]: there is again only one such regular solution, namely Minkowski space-time. It is likewise of interest to enquire what happens when  $\Lambda < 0$ . In this work we show that, similarly, no Lichnerowicz-type theorem exists in this case: we prove existence of infinite-dimensional families of static metrics satisfying the Einstein-Maxwell equations with a negative cosmological constant and admitting a smooth conformal completion at infinity.

More precisely, we show that a large class of such fields can be constructed by prescribing the conformal class of a static Lorentzian metric and the asymptotic behaviour of the electric field on the conformal boundary  $\partial\mathcal{M}$ , provided that the boundary class is sufficiently close to, e.g., that of anti-de Sitter spacetime and the freely prescribable leading coefficient in an asymptotic expansion of the electric potential is sufficiently small. This complements our previous work on vacuum metrics in [8], which we think of as paper I in this series.

The key new feature of the current result, as compared to [3, 4, 8], is that we can drive the solution with the electric field maintaining, if desired, a conformally flat structure at the conformal boundary at infinity.

Once this work was completed we have noticed [7], where a class of numerical solutions of the Einstein-Maxwell equations with  $\Lambda < 0$  is presented. Our work justifies rigorously the existence of weak-matter-fields configurations within the family constructed numerically in [7], and provides many other static non-vacuum solutions near anti-de Sitter.

We thus seek to construct Lorentzian metrics  $\mathbf{g}$ , solutions of Einstein-Maxwell equations with a negative cosmological constant  $\Lambda$ , in any space-dimension  $n \geq 3$ . More precisely, we consider the following field equations for a metric

$$\mathbf{g} = g_{\mu\nu} dx^\mu dx^\nu$$

and a two-form field  $F$  in space-time dimension  $n + 1$ ,  $n \geq 3$  generalising the Einstein-Maxwell equations in dimension  $3 + 1$  as

$$(1.1) \quad \text{Ric}(\mathbf{g}) - \frac{\text{Tr Ric}(\mathbf{g})}{2} \mathbf{g} + \Lambda \mathbf{g} = F \circ F - \frac{1}{4} \langle F, F \rangle_{\mathbf{g}} \mathbf{g},$$

where  $\text{Tr}$  denotes a trace with the relevant metric (which should be obvious from the context),  $(F \circ F)_{\alpha\beta} := g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}$ , and

$$\langle F, F \rangle \equiv g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} =: |F|^2.$$

This is complemented with the Maxwell equations

$$(1.2) \quad \text{div}_{\mathbf{g}} F = 0 = dF.$$

We further assume existence of a hypersurface-orthogonal globally timelike Killing vector  $X = \partial/\partial t$ . In adapted coordinates the space-time metric can

be written as

$$(1.3) \quad \mathbf{g} = -V^2 dt^2 + \underbrace{g_{ij} dx^i dx^j}_{=g},$$

$$(1.4) \quad \partial_t V = \partial_t g = 0.$$

We further assume that the Maxwell field takes the form

$$(1.5) \quad F = d(U dt), \quad \partial_t U = 0.$$

Our main result reads as follows (see Section 2 below for the definition of non-degeneracy; the function  $\rho$  in (1.7) is a coordinate near the conformal boundary at infinity  $\partial M$  that vanishes at  $\partial M$ ):

**THEOREM 1.1.** *Let  $n = \dim M \geq 3$ ,  $k \in \mathbb{N} \setminus \{0\}$ ,  $\alpha \in (0, 1)$ , and consider an Einstein metric  $\mathring{\mathbf{g}}$  as in (1.3)-(1.4) with strictly positive  $V = \mathring{V}$ ,  $g = \mathring{g}$ , such that the associated Riemannian metric  $\mathring{\mathbf{g}} = \mathring{V}^2 d\varphi^2 + \mathring{g}$  on  $\mathbb{S}^1 \times M$  is  $C^2$ -compactifiable and non-degenerate, with smooth conformal infinity. For every smooth  $\widehat{U}$ , sufficiently close to zero in  $C^{k+2, \alpha}(\partial M)$ , there exists a unique, modulo diffeomorphisms which are the identity at the boundary, solution of the static Einstein-Maxwell equations of the form (1.3)-(1.5) such that, in local coordinates near the conformal boundary  $\partial M$ ,*

$$(1.6) \quad V - \mathring{V} = O(\rho), \quad g_{ij} - \mathring{g}_{ij} = O(1),$$

$$(1.7) \quad U = \widehat{U} + \begin{cases} O(\rho), & n = 3; \\ O(\rho^2 \ln \rho), & n = 4; \\ O(\rho^2), & n \geq 5. \end{cases}$$

**REMARK 1.2.** We have emphasised the freedom to choose the leading-order behaviour  $\widehat{U}$  of  $U$ . As such, the leading-order behaviour of both  $\mathring{V}$  and  $\mathring{g}$  is, similarly, freely prescribable near a non-degenerate solution [4, 8]. Here one can, if desired, proceed in two steps: first, find the static solution with new fields  $\mathring{V}$  and  $\mathring{g}$  near a non-degenerate solution; small such perturbations will preserve non-degeneracy. One can then use Theorem 1.1 at the perturbed static solution to obtain a solution with the desired small  $\widehat{U}$ .  $\square$

The  $(n + 1)$ -dimensional anti-de Sitter metric is non-degenerate in the sense above (see eg. [4] appendix D), so Theorem 1.1 provides in particular an infinite dimensional family of solutions near that metric. We note existence of further large classes of Einstein metrics satisfying the non-degeneracy condition [1, 2, 4, 15].

The requirement of strict positivity of  $\mathring{V}$  excludes black hole solutions, we are hoping to return to the construction of black-hole solutions in a near future. In fact, this work was prompted by [13], where static Einstein-Maxwell black holes solutions driven by the asymptotics of the electric field have been constructed numerically.

Similarly to [13], for generic  $\widehat{U}$  the resulting space-time metric will have no isometries other than time-translations.

When the free boundary data  $\widehat{U}$  are smooth and when  $\mathring{\mathbf{g}}$  is conformally smooth, the solutions constructed here will have a polyhomogeneous expansion at the conformal boundary at infinity. The proof of this is an immediate repetition of the argument presented in [8, Section 7], where the reader

can also find the definition of polyhomogeneity. The decay rates in (1.6)-(1.7) have to be compared with the local-coordinates leading-order behavior  $\rho^{-2}$  both for  $\mathring{V}^2$  and  $\mathring{g}_{ij}$ . A more precise version of (1.6)-(1.7) in terms of weighted function spaces (we follow the notation in [8, 15]) reads, in all dimensions,

$$(1.8) \quad (V - \mathring{V}) \in C_1^{k+2,\alpha}(M), \quad (g - \mathring{g}) \in C_2^{k+2,\alpha}(M, \mathcal{S}_2),$$

$$(1.9) \quad U - \widehat{U} \in C_1^{k+2,\alpha}(M),$$

and the norms of the differences above are small in those spaces. To this one can add, in dimensions  $n \geq 4$ ,

$$(1.10) \quad U - \widehat{U} - U_{\text{ln}} \rho^2 \ln \rho \in C_2^{k+2,\alpha}(M),$$

with the function  $U_{\text{ln}} \in C^\infty(\mathbb{S}^1 \times \overline{M})$  given by (5.4) below when  $n = 4$ , and  $U_{\text{ln}} \equiv 0$  in dimension  $n \geq 5$ . Interestingly enough, in dimension  $n = 4$  the function  $U_{\text{ln}}$  vanishes only when  $dU \equiv 0$ , in which case the space-time is vacuum, see Remark 5.3 below.

The proof of Theorem 1.1 can be found at the end of Section 5. It follows closely [8], and proceeds through an implicit-function argument, with the key isomorphism properties of the associated linearised operator borrowed from [8, 15], and presented in Section 3.

As already noted, a constant  $\widehat{U}$  implies vacuum, by uniqueness of solutions. Hence the energy density of the Maxwell field behaves as  $\rho^4$  for small  $\rho$  if not identically zero (cf. (A.14) below), which leads to finite-total-energy configurations in space-dimensions  $n = 3$  and 4, but infinite matter energy in higher dimensions.

## 2. DEFINITIONS, NOTATIONS AND CONVENTIONS

Our definitions and conventions are identical to those in [8, Section 2]. Here we simply recall that the Lichnerowicz Laplacian acting on a symmetric two-tensor field is defined as [6, § 1.143]

$$\Delta_L h_{ij} = -\nabla^k \nabla_k h_{ij} + R_{ik} h^k{}_j + R_{jk} h^k{}_i - 2R_{ikjl} h^{kl}.$$

The operator  $\Delta_L + 2n$  arises naturally when linearising the equation

$$(2.1) \quad \text{Ric}(\mathfrak{g}) = -n\mathfrak{g},$$

where  $\text{Ric}(\mathfrak{g})$  is the Ricci curvature of  $\mathfrak{g}$ , at a solution. We will say that a metric is *non-degenerate* if  $\Delta_L + 2n$  has no  $L^2$ -kernel.

## 3. ISOMORPHISM THEOREMS

In this section we recall two isomorphism results from [8, Section 3] and prove an isomorphism theorem on scalar fields, as needed in the remainder of this work. The reader might also want to consult the introductory comments in [8, Section 3], which remain valid in the current context.

**3.1. An isomorphism on two-tensors.** We will need [8, Corollary 3.2]:

**COROLLARY 3.1.** *Let  $(\mathbb{S}^1 \times M, V^2 d\varphi^2 + g)$  be an asymptotically hyperbolic non-degenerate Riemannian manifold with  $\partial_\varphi V = 0 = \partial_\varphi g$ . Consider the map*

$$(W, h) \mapsto (l(W, h), L(W, h)),$$

where

$$(3.1) \quad l(W, h) = V[(\nabla^* \nabla + 2n + V^{-1} \nabla^* \nabla V + V^{-2} |dV|^2)W + V^{-1} \nabla_j V \nabla^j W - V^{-1} \nabla^j V \nabla^k V h_{kj} + \langle \text{Hess}_g V, h \rangle_g],$$

and

$$(3.2) \quad \begin{aligned} L_{ij}(W, h) &= \frac{1}{2} \Delta_L h_{ij} + n h_{ij} - \frac{1}{2} V^{-1} \nabla^k V \nabla_k h_{ij} \\ &\quad + \frac{1}{2} V^{-2} (\nabla_i V \nabla^k V h_{kj} + \nabla_j V \nabla^k V h_{ki}) \\ &\quad - \frac{1}{2} V^{-1} (\nabla_i \nabla^k V h_{kj} + \nabla_j \nabla^k V h_{ki}) \\ &\quad + 2V^{-2} W (\text{Hess}_g V)_{ij} - 2V^{-3} \nabla_i V \nabla_j V W. \end{aligned}$$

Then  $(l, L)$  is an isomorphism from  $C_{\delta-1}^{k+2, \alpha}(M) \times C_\delta^{k+2, \alpha}(M, \mathcal{S}_2)$  to  $C_{\delta-2}^{k, \alpha}(M) \times C_\delta^{k, \alpha}(M, \mathcal{S}_2)$  when  $\delta \in (0, n)$ .

**3.2. An isomorphism on one-forms.** We will also need the following sharper version of [8, Corollary 3.5], where we note that the isomorphism given in Theorem 3.4 and Corollary 3.5 of [8] are in fact valid for a larger interval of weights, as can be established by a more careful inspection of the arguments presented in [8]:

**COROLLARY 3.2.** *Let  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$ . Under the hypotheses of Corollary 3.1, suppose moreover that the Ricci tensor of  $V^2 d\varphi^2 + g$  is negative. Consider the operator*

$$\Omega_i \mapsto B(\Omega)_i + R_{ij} \Omega^j - V^{-1} \nabla_i \nabla^j V \Omega_j =: \mathcal{B}(\Omega)_i,$$

where

$$B(\Omega)_i := \nabla^k \nabla_k \Omega_i + V^{-1} \nabla^k V \nabla_k \Omega_i - V^{-2} \nabla_i V \nabla^k V \Omega_k.$$

Then  $\mathcal{B}$  is an isomorphism from  $C_\delta^{k+2, \alpha}(M, \mathcal{T}_1)$  to  $C_\delta^{k, \alpha}(M, \mathcal{T}_1)$  when

$$\left| \delta - \frac{n}{2} \right| < \frac{n+2}{2}.$$

**3.3. An isomorphism on functions.** If we assume that  $V^2 d\varphi^2 + g$  is a static asymptotically hyperbolic metric on  $\mathbb{S}^1 \times M$ , then it is easy to check that

$$(3.3) \quad V^{-2} |dV|^2 \rightarrow 1 \text{ and } V^{-1} \nabla^i \nabla_i V \rightarrow n$$

as the conformal boundary is approached. We will need an isomorphism property for the following operator acting on functions with  $s = -1$ ; in [8] the result has already been established with  $s = -3$  and  $s = 3$ , so for future reference it appears useful to consider all values of  $s$ :

$$\sigma \mapsto \mathcal{T}_s \sigma := V^{-s} \nabla^i (V^s \nabla_i \sigma) = \nabla^i \nabla_i \sigma + s V^{-1} \nabla^i V \nabla_i \sigma.$$

**THEOREM 3.3.** *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold with an asymptotically hyperbolic metric with  $V > 0$  and assume that (3.3) holds. Let  $s \neq 1 - n$  and suppose that*

$$\frac{s + n - 1 - |s + n - 1|}{2} < \delta < \frac{s + n - 1 + |s + n - 1|}{2}.$$

*If  $s \geq -\frac{n-1}{2}$  then  $\mathcal{T}_s$  is an isomorphism from  $C_\delta^{k+2, \alpha}(M)$  to  $C_\delta^{k, \alpha}(M)$ . If  $s < -\frac{n-1}{2}$ , then  $\mathcal{T}_s$  is an isomorphism from  $C_\delta^{k+2, \alpha}(M)/\mathbb{R}$  to*

$$(3.4) \quad \left\{ \sigma \in C_\delta^{k, \alpha}(M) : \int_M V^s \sigma = 0 \right\}.$$

*Proof.* When  $s + n - 1 > 0$ , we can use [5, Theorem 7.2.1 (ii) and Remark (i), p. 77] to conclude. As we want the result for any  $s \neq 1 - n$ , for the remaining cases we appeal to the results of Lee [15]. For this we need a formally self-adjoint operator, so we set  $\sigma = V^{-\frac{s}{2}} f$ , thus

$$(3.5) \quad \mathcal{T}_s \sigma = V^{-\frac{s}{2}} \left[ \nabla^i \nabla_i f - \frac{s}{2} \left( \left( \frac{s}{2} - 1 \right) V^{-2} |dV|^2 + V^{-1} \nabla^i \nabla_i V \right) f \right] =: V^{-\frac{s}{2}} T_s f.$$

By assumption  $V^{-2} |dV|^2 \rightarrow 1$  and  $V^{-1} \nabla^i \nabla_i V \rightarrow n$  at the conformal boundary, leading to the following indicial exponents for  $T_s$ :

$$\delta = \frac{n - 1 \pm |s + n - 1|}{2}.$$

We want to show that  $T_s$  satisfies condition (1.4) of [15],

$$(3.6) \quad \|u\|_{L^2} \leq C \|T_s u\|_{L^2},$$

for smooth  $u$  compactly supported in a sufficiently small open set  $\mathcal{U} \subset M$  such that  $\bar{\mathcal{U}}$  is a neighborhood of  $\partial M$ . Let us recall the following, well known result (see eg. [8, Lemma 3.8]):

**LEMMA 3.4.** *On an asymptotically hyperbolic manifold  $(M, g)$  with boundary defining function  $\rho$  we have, for all compactly supported  $C^2$  functions,*

$$\int u \nabla^* \nabla u \geq \left( \frac{n-1}{2} \right)^2 \int (1 + O(\rho)) u^2.$$

Lemma 3.4 combined with the hypothesis (3.3) shows that

$$\|u\|_{L^2} \|T_s u\|_{L^2} \geq - \int u T_s u \geq \int \left( \frac{(n-1)^2}{4} + \frac{s}{2} \left( \frac{s}{2} + n - 1 \right) + o(1) \right) u^2,$$

from which it follows that  $T_s$  satisfies indeed (3.6) with

$$C^{-1} = \frac{(s + n - 1)^2}{4}.$$

We recall that the critical fall-off for a function to belong to  $L^2$  is  $O(\rho^{\frac{n-1}{2}})$ . This can be used to show that the  $L^2$ -kernel of  $T_s$  equals

$$\ker T_s = V^{\frac{s}{2}} \mathbb{R} \cap L^2 = \begin{cases} \{0\} & \text{if } s \geq -\frac{n-1}{2}, \\ V^{\frac{s}{2}} \mathbb{R} & \text{if } s < -\frac{n-1}{2}. \end{cases}$$

Indeed, assume that  $f$  is in the  $L^2$ -kernel of  $T_s$ . By elliptic regularity (see [15, Lemma 4.8] for instance)  $f$  is  $H^2$  on  $M$ . This implies in particular that no boundary terms arise in the following integration by parts:

$$\begin{aligned} 0 &= - \int f T_s f = - \int f V^{-\frac{s}{2}} \nabla^i [V^s \nabla_i (V^{-\frac{s}{2}} f)] \\ &= \int V^s |\nabla (V^{-\frac{s}{2}} f)|^2. \end{aligned}$$

So  $V^{-\frac{s}{2}} f$  is a constant. Using [15], Theorem C(c), we find:

- If  $s \geq \frac{n-1}{2}$  then the kernel (and then the cokernel) of  $T_s$  is trivial so  $T_s$  is an isomorphism from  $C_{\delta^{-\frac{s}{2}}}^{k+2,\alpha}(M)$  to  $C_{\delta^{-\frac{s}{2}}}^{k,\alpha}(M)$ .
- If  $s < -\frac{n-1}{2}$ , then  $T_s$  is an isomorphism from  $C_{\delta^{-\frac{s}{2}}}^{k+2,\alpha}(M)/V^{s/2}\mathbb{R}$  to the orthogonal to the kernel:

$$\left\{ f \in C_{\delta^{-\frac{s}{2}}}^{k,\alpha}(M) : \int_M V^{s/2} f = 0 \right\}.$$

The conclusion follows for  $\mathcal{T}_s$  if we recall that  $\sigma = V^{-\frac{s}{2}} f$  is in  $C_{\delta}^{k,\alpha}(M)$  iff  $f \in C_{\delta^{-\frac{s}{2}}}^{k,\alpha}(M)$ .  $\square$

#### 4. THE EQUATIONS

Rescaling the metric to achieve a convenient normalisation of the cosmological constant,

$$(4.1) \quad \Lambda = -\frac{n(n-1)}{2},$$

the vacuum Einstein-Maxwell equations for a metric satisfying (1.3)-(1.4) read (see Appendix A)

$$(4.2) \quad \begin{cases} V(\nabla^* \nabla V + nV) = -\frac{n-2}{n-1} |dU|_g^2, \\ \text{Ric}(g) + ng - V^{-1} \text{Hess}_g V = V^{-2} \left( -dU \otimes dU + \frac{1}{n-1} |dU|_g^2 g \right), \\ \text{div}_g(V^{-1} \nabla U) = 0. \end{cases}$$

**4.1. The linearised equation.** We use the symbol  $\text{Tr}$ , or  $\text{Tr}_g$  when the metric could be ambiguous, to denote the trace. As in [8] we set

$$\text{grav } h = h - \frac{1}{2} \text{Tr}_g h g, \quad (\text{div } h)_i = -\nabla^k h_{ik}, \quad (\text{div}^* w)_{ij} = \frac{1}{2} (\nabla_i w_j + \nabla_j w_i)$$

(note the geometers' convention, in which we have a negative sign in the definition of divergence). We consider the operator from the set of functions times symmetric two tensor fields to itself, defined as

$$\begin{pmatrix} V \\ g \end{pmatrix} \mapsto \begin{pmatrix} V(\nabla^* \nabla V + nV) \\ \text{Ric}(g) + ng - V^{-1} \text{Hess}_g V \end{pmatrix}.$$



(To avoid ambiguities,  $\nabla^* \nabla \equiv -\nabla^i \nabla_i =: -\Delta$ .) The two components of its linearisation at  $(V, g)$  are

$$\begin{aligned} p(W, h) &= V \left[ (\nabla^* \nabla + 2n + V^{-1} \nabla^* \nabla V) W + \langle \text{Hess}_g V, h \rangle_g \right. \\ &\quad \left. - \langle \text{div grav } h, dV \rangle_g \right], \\ P_{ij}(W, h) &= \frac{1}{2} \Delta_L h_{ij} + n h_{ij} + \frac{1}{2} V^{-1} \nabla^k V (\nabla_i h_{kj} + \nabla_j h_{ki} - \nabla_k h_{ij}) \\ &\quad - (\text{div}^* \text{div grav } h)_{ij} + V^{-2} W (\text{Hess}_g V)_{ij} - V^{-1} (\text{Hess}_g W)_{ij}. \end{aligned}$$

It turns out to be convenient to introduce the one-form

$$w_j = V^{-1} \nabla^k V h_{kj} + \nabla^k h_{kj} - \frac{1}{2} \nabla_j (\text{Tr } h) - V^{-1} \nabla_j W - V^{-2} \nabla_j V W,$$

which allows us to rewrite  $P(W, h)$  as

$$P(W, h) = L(W, h) + \text{div}^* w,$$

where  $L$  is as in Corollary 3.1. Similarly,  $p(W, h)$  can be rewritten as

$$p(W, h) = l(W, h) + V \langle w, dV \rangle_g,$$

with  $l$  given by (3.1).

**4.2. The modified equation.** In Section 5 we will use the implicit function theorem to construct our solutions, using the observation of [11], how to obtain a well-behaved equation by adding ‘‘gauge fixing terms’’. We choose those terms as in [8], appealing to harmonic maps for the vacuum Einstein equations in one dimension higher.

Indeed, we will be solving the following system of equations

$$(4.3) \quad \begin{cases} V(\nabla^* \nabla V + nV + \langle \Omega, dV \rangle) + \frac{n-2}{n-1} |dU|_g^2 = 0, \\ \text{Ric}(g) + ng - V^{-1} \text{Hess}_g V + \text{div}^* \Omega \\ \quad - V^{-2} \left( -dU \otimes dU + \frac{1}{n-1} |dU|_g^2 g \right) = 0, \\ \text{div}_g(V^{-1} \nabla U) = 0, \end{cases}$$

where

$$(4.4) \quad \begin{aligned} -\Omega_j &\equiv -\Omega(V, g, \mathring{V}, \mathring{g})_j \\ &:= \mathfrak{g}_{j\mu} \mathfrak{g}^{\alpha\beta} (\Gamma(\mathfrak{g})_{\alpha\beta}^\mu - \Gamma(\mathring{\mathfrak{g}})_{\alpha\beta}^\mu) \\ &= g^{\ell m} (\mathring{\nabla}_m g_{j\ell} - \frac{1}{2} \mathring{\nabla}_j g_{\ell m}) + V^{-2} g_{jk} (\mathring{V} \mathring{\nabla}^k \mathring{V} - V \nabla^k V), \end{aligned}$$

and where  $\mathring{\nabla}$ -derivatives are relative to the metric  $\mathring{g}$ , with Christoffel symbols  $\mathring{\Gamma}_{\beta\gamma}^\alpha$ , Latin indices run from 0 to  $n$ , and  $\mathfrak{g} := V^2(dx^0)^2 + g$  with Christoffel symbols  $\Gamma(\mathfrak{g})_{\beta\gamma}^\alpha$ , while the  $\Gamma(\mathring{\mathfrak{g}})_{\beta\gamma}^\alpha$ 's are the Christoffel symbols of the metric  $\mathring{\mathfrak{g}}$ .

The derivative of  $\Omega$  with respect to  $(V, g)$  at  $(\mathring{V}, \mathring{g})$  is

$$D_{(V, g)} \Omega(\mathring{V}, \mathring{g}, \mathring{V}, \mathring{g})(W, h) = -w,$$

where  $w$  is the one-form defined in Section 4.1 with  $(V, g)$  replaced with  $(\mathring{V}, \mathring{g})$ . Thus, the linearisation of  $(q, Q)$  at  $(\mathring{V}, \mathring{g})$  is

$$D(q, Q)(\mathring{V}, \mathring{g}) = (l, L),$$

where  $(l, L)$  is the operator defined in Section 4.1 with  $(V, g)$  replaced with  $(\mathring{V}, \mathring{g})$ . We will show that, under reasonable conditions, solutions of (4.3) are solutions of (4.2).

**PROPOSITION 4.1.** *If  $(U, V, g)$  solves (4.3) then  $\Omega$  is in the kernel of  $\mathcal{B}$ . If moreover  $\Omega \in C_\delta^{2,\alpha}$  for some  $\delta > -1$ , then  $\Omega \equiv 0$ , so that  $(U, V, g)$  solves (4.2).*

*Proof.* A standard adaptation of the argument in [8] gives the result; we sketch this for completeness. Set

$$\begin{aligned} a &:= -\frac{n-2}{n-1}V^{-2}|dU|^2, \\ A &:= V^{-2}(-dU \otimes dU + \frac{1}{n-1}|dU|^2g). \end{aligned}$$

With this notation, the first two equations in (4.3) take the form

$$(4.5) \quad \begin{cases} \nabla^* \nabla V + nV + \langle \Omega, dV \rangle = Va, \\ \text{Ric}(g) + ng - V^{-1} \text{Hess}_g V + \text{div}^* \Omega = A, \end{cases}$$

The tensor field  $E(g) = \text{grav}_g \text{Ric}(g)$ , has vanishing divergence, which provides the equation  $\mathcal{B}(\Omega) = 0$  for  $\Omega$ . Indeed, the operator  $\mathcal{B}(\Omega)$  defined above has been constructed in [8] so that it vanishes when the modified equation for the metric holds and when the Lorentzian  $(n+1)$ -dimensional divergence of the  $(n+1)$ -dimensional energy-momentum tensor vanishes. The latter condition is satisfied when the  $(n+1)$ -dimensional matter field equations hold; in the current case, this coincides with the last equation in (4.3).

An alternative,  $n$ -dimensional argument proceeds as follows: The calculation following [8, Equation (4.6)] shows that for a solution to the modified equation, the divergence of  $E(g)$  equals

$$0 \equiv \text{div} E(g) = \frac{1}{2} \mathcal{B}(\Omega) + \beta,$$

where

$$(4.6) \quad \beta := V^{-1}d(Va) - V^{-1}A(\nabla V, \cdot) + \text{div}(\text{grav}_g A + \frac{a}{2}g).$$

The vanishing of  $\beta$  can be checked by a somewhat lengthy calculation using (4.3).

Either way, the Bianchi identity  $\text{div} E(g) = 0$  shows that  $\Omega$  is in the kernel of  $\mathcal{B}$ . It follows from Corollary 3.2 that the only solution of this equation which decays as described is zero.  $\square$

For future reference we consider the static equations, modified as in (4.5), with a general energy-momentum tensor:

$$(4.7) \quad -V^{-1} \nabla^i \nabla_i V - \frac{2\Lambda}{n-1} + V^{-1} \nabla^i \Omega \nabla_i V = \underbrace{-T_{\alpha\beta} N^\alpha N^\beta - \frac{g^{\alpha\beta} T_{\alpha\beta}}{n-1}}_{=:a},$$

$$(4.8) \quad R_{ij} - V^{-1} \nabla_i \nabla_j V - \frac{2\Lambda}{n-1} g_{ij} + \frac{1}{2} (\nabla_i \Omega_j + \nabla_j \Omega_i) = \underbrace{T_{ij} - \frac{g^{\alpha\beta} T_{\alpha\beta}}{n-1} g_{ij}}_{=:A_{ij}},$$

where  $N^\alpha \partial_\alpha$  is the unit timelike normal to the initial data surface (compare (A.8)-(A.9), Appendix A). We have:

PROPOSITION 4.2. *Let  $T_{\mu\nu}$  be divergence-free with respect to the time-independent metric*

$$\mathbf{g} = -V^2 dt^2 + g$$

as a consequence of the matter field equations, with  $\partial_t T_{\mu\nu} = T_{0i} = 0$ . Set

$$(4.9) \quad a := -T_{\alpha\beta} N^\alpha N^\beta - \frac{g^{\alpha\beta} T_{\alpha\beta}}{n-1}, \quad A_{ij} := T_{ij} - \frac{g^{\alpha\beta} T_{\alpha\beta}}{n-1} g_{ij}.$$

Then

$$(4.10) \quad \beta \equiv 0.$$

*Proof.* Equation (4.6) can be rewritten as

$$(4.11) \quad \begin{aligned} -\beta_j &= \nabla_i \left( A^i_j - \frac{1}{2} (A^k_k + a) \delta_j^i \right) + V^{-1} (A_{ij} - a g_{ij}) \nabla V^i \\ &= \nabla_i \left( T^i_j - \frac{1}{2} \left( \frac{2g^{\alpha\beta} T_{\alpha\beta}}{n-1} + A^k_k + a \right) \delta_j^i \right) \\ &\quad + V^{-1} \left( T_{ij} - \left( \frac{g^{\alpha\beta} T_{\alpha\beta}}{n-1} - T_{\alpha\beta} N^\alpha N^\beta - \frac{g^{\alpha\beta} T_{\alpha\beta}}{n-1} \right) g_{ij} \right) \nabla V^i. \end{aligned}$$

We have

$$\begin{aligned} &\frac{2g^{\alpha\beta} T_{\alpha\beta}}{n-1} + A^k_k + a \\ &= \frac{2g^{\alpha\beta} T_{\alpha\beta}}{n-1} + g^{ij} T_{ij} - \frac{n}{n-1} g^{\alpha\beta} T_{\alpha\beta} - T_{\alpha\beta} N^\alpha N^\beta - \frac{g^{\alpha\beta} T_{\alpha\beta}}{n-1} = 0. \end{aligned}$$

The  $(n+1)$ -decomposition of the equation  $\nabla^\mu T_{\mu\nu} = 0$ , where  $\nabla_\mu$  is the space-time covariant derivative associated with  $\mathbf{g}$ , using the formulae for the Christoffel symbols in Appendix A gives

$$(4.12) \quad \nabla^i T_{ij} + V T_{\alpha\beta} N^\alpha N^\beta \nabla_j V + V^{-1} \nabla^j V T_{ij} = 0,$$

where  $\nabla^i$  is the covariant derivative operator of  $g$ . Comparing with (4.11) gives the result.  $\square$

## 5. THE CONSTRUCTION

We consider an asymptotically hyperbolic Einstein static metric  $\mathring{\mathbf{g}} = \mathring{V}^2 d\varphi^2 + \mathring{g}$  on  $\mathbb{S}^1 \times M$ . We apply Theorem 3.3 with  $s = -1$ ,  $g$  — a Riemannian metric on  $M$  close to  $\mathring{g}$  in  $C_0^{k+2,\alpha}(M, \mathcal{S}_2)$ , and  $V$  — a function on  $M$  close to  $\mathring{V}$  in  $C_{-1}^{k+2,\alpha}(M)$ . It is convenient to choose some

$$(5.1) \quad \delta \in (0, 1) \text{ when } n = 3 \text{ and } \delta = 1 \text{ if } n > 3.$$

We conclude that for any  $\widehat{U} \in C^{k+2,\alpha}(\partial M)$ , there exists a unique solution

$$U = U(\widehat{U}, V, g) \in C_0^{k+2,\alpha}(M)$$

to

$$\begin{cases} \nabla^*(V^{-1} \nabla U) = 0, \\ U - \widehat{U} \in C_\delta^{k+2,\alpha}(M). \end{cases}$$

Moreover, the map  $(\widehat{U}, V, g) \mapsto U - \widehat{U} \in C_\delta^{k+2,\alpha}(M)$  is smooth.

We define a new map  $F$ , defined on the set of functions on the conformal boundary at infinity  $\partial_\infty M$  times functions on  $M$  times symmetric two-tensor

fields on  $M$ , mapping to functions on  $M$  times symmetric two-tensor fields on  $M$ , which to  $(\widehat{U}, V, g)$  associates

$$\left( \begin{array}{c} V(\nabla^* \nabla V + nV + \langle \Omega(V, g, \mathring{V}, \mathring{g}), dV \rangle) + \frac{n-2}{n-1} |dU|^2 \\ \text{Ric}(g) + ng - V^{-1} \nabla_i \nabla_j V + \text{div}^* \Omega(V, g, \mathring{V}, \mathring{g}) + V^{-2} (dU \otimes dU - \frac{1}{n-1} |dU|^2 g) \end{array} \right).$$

PROPOSITION 5.1. *Let  $\mathring{g} = \mathring{V}^2 d\varphi^2 + \mathring{g}$  be an asymptotically hyperbolic static Einstein metric on  $\mathbb{S}^1 \times M$ ,  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$ . The map  $\mathcal{F}$  defined as*

$$\begin{array}{ccc} C^{k+2, \alpha}(\partial M) \times C_1^{k+2, \alpha}(M) \times C_2^{k+2, \alpha}(M, \mathcal{S}_2) & \longrightarrow & C_0^{k, \alpha}(M) \times C_2^{k, \alpha}(M, \mathcal{S}_2) \\ (\widehat{U}, W, h) & \longmapsto & F(\widehat{U}, \mathring{V} + W, \mathring{g} + h) \end{array}$$

is smooth in a neighborhood of zero.

*Proof.* Let  $\delta$  be as in (5.1). We have, by direct estimations,

$$V^{-2} (dU \otimes dU - \frac{1}{n-1} |dU|^2 g) \in C_{2+2\delta}^{k, \alpha}(M, \mathcal{S}_2) \subset C_2^{k, \alpha}(M, \mathcal{S}_2).$$

The remaining arguments of the proof of [8, Proposition 5.2] apply with trivial modifications: The function  $\mathring{V} \in C_{-1}^{k+2, \alpha}(M)$  is strictly positive, so the same is true for  $\mathring{V} + W$  if  $W$  is sufficiently small in  $C_1^{k+2, \alpha}(M) \subset C_{-1}^{k+2, \alpha}(M)$ . Similarly, the symmetric two-tensor field  $\mathring{g} + h \in C_0^{k+2, \alpha}(M, \mathcal{S}_2)$  is positive definite when  $h$  is small in  $C_2^{k+2, \alpha}(M, \mathcal{S}_2) \subset C_0^{k+2, \alpha}(M, \mathcal{S}_2)$ . The map  $(\widehat{U}, V, g) \mapsto U$  is smooth. The fact that the remaining terms in  $F(\widehat{U}, \mathring{V} + W, \mathring{g} + h)$  are in the space claimed, and that the map is smooth, follows from standard calculations (see [12, proof of Theorem 4.1] or [10, Appendix] for associated detailed calculations).  $\square$

We are ready to formulate now:

THEOREM 5.2. *Let  $\dim M = n \geq 3$  and let  $\mathring{V}^2 d\varphi^2 + \mathring{g}$  be a non-degenerate asymptotically hyperbolic static Einstein metric on  $\mathbb{S}^1 \times M$ ,  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$ ,  $\delta \in (0, 1)$  in  $n = 3$  and  $\delta = 1$  if  $n > 3$ . For all  $\widehat{U}$  close to zero in  $C^{k+2, \alpha}(\partial M)$  there exists a unique solution*

$$(U, V, g) = (\widehat{U} + u, \mathring{V} + W, \mathring{g} + h)$$

to (4.2) with

$$(u, W, h) \in C_\delta^{k+2, \alpha}(M) \times C_1^{k+2, \alpha}(M) \times C_2^{k+2, \alpha}(M, \mathcal{S}_2),$$

close to zero, satisfying the gauge condition  $\Omega = 0$ . Moreover, the map  $\widehat{U} \mapsto (u, W, h)$  is a smooth map of Banach spaces near zero.

*Proof.* As already pointed out, the function  $U = U(\widehat{U}, V, g)$  exists and is unique when  $W$  and  $h$  are small. From Proposition 5.1 we know that the map  $\mathcal{F}$  is smooth. The linearisation of  $\mathcal{F}$  at zero is

$$D_{(W, h)} \mathcal{F}(0, 0, 0) = D_{(V, g)} F(0, \mathring{V}, \mathring{g}) = (l, L).$$

From Corollary 3.1, with  $\delta = 2$ , we obtain that  $D_{(W, h)} \mathcal{F}(0, 0, 0)$  is an isomorphism. The implicit function theorem shows that the conclusion of Theorem 5.2 remains valid for the modified equation (4.3). Returning to Section 4.2, we see that  $\Omega = \Omega(V, g, \mathring{V}, \mathring{g}) \in C_2^{k+1, \alpha}(M, T_1)$ . Corollary 3.2 gives  $\Omega = 0$ , and we have obtained a solution to (4.2).  $\square$

PROOF OF THEOREM 1.1: Existence and uniqueness of a solution with

$$(5.2) \quad V - \mathring{V} = O(\rho), \quad U = \widehat{U} + O(\rho^\delta), \quad g - \mathring{g} = O_{\mathring{g}}(\rho^2) = O_{\rho^2 \mathring{g}}(1),$$

follows from Theorem 5.2. Here and elsewhere we write  $u = O(\rho^\sigma)$  for a tensor  $u$  if the coordinate components of  $u$  in local coordinates near the boundary are  $O(\rho^\sigma)$ , and  $u = O_{\mathring{g}}(\rho^\sigma)$  if the norm  $|u|_{\mathring{g}}$  of  $u$  with respect to the metric  $\mathring{g}$  is  $O(\rho^\sigma)$ . Standard arguments show that all the fields  $V$ ,  $U$  and  $g$  are polyhomogeneous (cf. e.g. [9], compare [8, Section 7]).

To justify (1.7), we plug-in a polyhomogeneous asymptotics as in (5.2) in the equations. Since the trace-free part of the Ricci tensor of  $\mathbf{g} = -V^2 dt^2 + g$  decays in  $\mathring{g}$ -norm as  $\rho^4$  near  $\rho = 0$ , the metric  $\mathbf{g} = -V^2 dt^2 + g$  is vacuum up to this order, which implies that the usual Fefferman-Graham expansion holds for  $\mathbf{g}$  up to terms  $O_{\mathring{g}}(\rho^4 \ln \rho)$ . This leads to the following form of the metric near the conformal boundary

$$(5.3) \quad \mathring{g} = \rho^{-2}(d\rho^2 + \mathring{h}), \quad \mathring{h} = \check{h} + O_{\check{h}}(\rho^2), \quad \mathring{V} = \check{V}\rho^{-1} + O(\rho),$$

where  $\check{h}$  is a family of Riemannian metrics on the boundary depending upon  $\rho$ , with

$$g = \mathring{g} + O_{\mathring{g}}(\rho^2), \quad V = \mathring{V} + O(\rho).$$

The equation  $\nabla_i(V^{-1}\nabla^i U) = 0$  in local coordinates  $(\rho, x^A)$  as in (5.3) takes the form

$$\begin{aligned} & \check{V}^{-1}\sqrt{\det \check{h}} \partial_\rho [\rho^{3-n}(1 + O(\rho^2))\partial_\rho U] \\ &= -\rho^{3-n}\partial_A \left[ \left( \check{V}^{-1}\sqrt{\det \check{h}} + O(\rho^2) \right) (\check{h}^{AB} + O^{AB}(\rho^2))\partial_B U \right] \\ & \quad + \partial_\rho (\rho^{5-n}O^A(1)\partial_A U) + \partial_A (\rho^{5-n}O^A(1)\partial_\rho U). \end{aligned}$$

Let  $U$  be a polyhomogeneous solution with  $U = \widehat{U} + O(\rho^\delta)$ . When  $n = 3$ , we can see that no logarithmic term  $O(\rho \ln \rho)$  is needed in  $U$  so that  $U = \widehat{U} + O(\rho)$ . When  $n = 4$ , we find

$$(5.4) \quad U = \widehat{U} - \underbrace{\frac{1}{2}\check{V}\check{\nabla}_A(\check{V}^{-1}\check{\nabla}^A\widehat{U})}_{=:U_{\text{ln}}}\rho^2 \ln \rho + O(\rho^2).$$

(In dimensions  $n \geq 5$  one expects in general a logarithmic term  $O(\rho^{n-2} \ln \rho)$  in an asymptotic expansion of  $U$ .)  $\square$

REMARK 5.3. The equality

$$2 \int_{\partial M} \check{V}^{-1}\widehat{U}U_{\text{ln}} = \int_{\partial M} \check{V}^{-1}|\check{\nabla}\widehat{U}|^2,$$

together with the definition of  $U_{\text{ln}}$ , proves that  $U_{\text{ln}} = 0$  if and only if  $\widehat{U}$  is constant. When  $\widehat{U}$  is constant the function  $U := \widehat{U}$  solves the equations and satisfies the right-boundary values, so the associated solution is vacuum.

APPENDIX A. STATIC EINSTEIN-MAXWELL EQUATIONS IN DIMENSIONS  
 $n + 1 \geq 4$ 

Consider a Riemannian manifold  $(M, g)$ , let us denote by  $\nabla$  the Levi-Civita derivative operator associated with  $g$ . Let  $V : M \rightarrow \mathbb{R}$ , and set

$$(\mathcal{M} = \mathbb{R} \times M, \tilde{g} = \varepsilon V^2 dt^2 + g), \quad \varepsilon = \pm 1,$$

and  $(x^\mu) = (x^0 = t, x^i = (x^1, \dots, x^n))$ . We have

$$(A.1) \quad \tilde{\Gamma}_{00}^0 = \tilde{\Gamma}_{ij}^0 = \tilde{\Gamma}_{i0}^k = 0, \quad \tilde{\Gamma}_{i0}^0 = V^{-1} \partial_i V, \quad \tilde{\Gamma}_{00}^k = -\varepsilon V \nabla^k V, \quad \tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k,$$

$$(A.2) \quad \tilde{R}^l{}_{ijk} = R^l{}_{ijk}, \quad \tilde{R}^l{}_{0j0} = -\varepsilon V \nabla_j \nabla^l V, \quad \tilde{R}^0{}_{ij0} = V^{-1} \nabla_j \nabla_i V,$$

$$(A.3) \quad \tilde{R}^0{}_{ijk} = \tilde{R}^l{}_{ij0} = \tilde{R}^l{}_{0jk} = 0,$$

$$(A.4) \quad \tilde{R}_{ik} = R_{ik} - V^{-1} \nabla_k \nabla_i V, \quad \tilde{R}_{0k} = 0, \quad \tilde{R}_{00} = -\varepsilon V \nabla^i \nabla_i V,$$

$$(A.5) \quad \tilde{R} = R - 2V^{-1} \nabla^i \nabla_i V.$$

Consider the  $(n + 1)$ -dimensional Einstein equations

$$(A.6) \quad \tilde{R}_{\alpha\beta} - \frac{1}{2} \tilde{R} \tilde{g}_{\alpha\beta} + \Lambda \tilde{g}_{\alpha\beta} = T_{\alpha\beta}.$$

Equations (A.6) with  $\tilde{g} = -V^2 dt^2 + g$  lead to

$$(A.7) \quad \tilde{R}_{\alpha\beta} = \frac{2\Lambda - \text{Tr}_{\tilde{g}} T}{n-1} \tilde{g}_{\alpha\beta} + T_{\alpha\beta},$$

$$(A.8) \quad R_{ij} = V^{-1} \nabla_i \nabla_j V + \frac{2\Lambda}{n-1} g_{ij} + T_{ij} - \frac{\text{Tr}_{\tilde{g}} T}{n-1} g_{ij},$$

$$(A.9) \quad V \nabla^i \nabla_i V = V^2 \left( T_{\alpha\beta} N^\alpha N^\beta + \frac{\text{Tr}_{\tilde{g}} T - 2\Lambda}{n-1} \right),$$

where  $N^\alpha \partial_\alpha$  is the unit timelike normal to the level sets of  $t$ .

For a static electric field  $F = d(Udt)$ ,  $\partial_t U = 0$ , and  $T_{\mu\nu}$  as in (1.1) we find

$$(A.10) \quad |F|^2 = -2V^{-2} |dU|^2,$$

$$(A.11) \quad \tilde{g}^{\mu\nu} T_{\mu\nu} \equiv \text{Tr}_{\tilde{g}} T = \frac{(3-n)}{4} |F|^2 = -\frac{(3-n)}{2} V^{-2} |dU|^2,$$

$$(A.12) \quad T_{ij} = -V^{-2} \nabla_i U \nabla_j U + \frac{1}{2} V^{-2} |dU|^2 g_{ij},$$

$$(A.13) \quad g^{ij} T_{ij} = \frac{(n-2)}{2} V^{-2} |dU|^2$$

$$(A.14) \quad T_{\widehat{00}} := T_{\mu\nu} N^\mu N^\nu = \frac{1}{2} V^{-2} |dU|^2.$$

Choosing  $\Lambda$  as in (4.1), Equation (A.9) gives

$$(A.15) \quad V \Delta V = \frac{n-2}{(n-1)} |dU|^2 - \frac{2\Lambda V^2}{n-1} = \frac{n-2}{(n-1)} |dU|^2 + nV^2.$$

When  $\tilde{g} = \varepsilon V^2 dt^2 + g$  is static and

$$(A.16) \quad T = T_{00} (dx^0)^2 + T_{ij} dx^i dx^j, \quad \partial_0 T_{00} = \partial_0 T_{ij} = 0,$$

then

$$(A.17) \quad \tilde{\nabla}^\alpha T_{\alpha\beta} dx^\beta = (\nabla^i T_{ij} + V^{-1} \nabla^i V T_{ij} - \varepsilon V^{-3} \nabla_j V T_{00}) dx^j.$$

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