Lp almost conformal isometries of Sub-Semi-Riemannian metrics and solvability of a Ricci equation

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L^p ALMOST CONFORMAL ISOMETRIES OF SUB-SEMI-RIEMANNIAN METRICS AND SOLVABILITY OF A RICCI EQUATION

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ABSTRACT. Let M be a smooth compact manifold without boundary. We consider two smooth Sub-Semi-Riemannian metrics on M. Under suitable conditions, we show that they are almost conformally isometric in an L^p sense. Assume also that M carries a Riemannian metric with parallel Ricci curvature. Then an equation of Ricci type, is in some sense solvable, without assuming any closeness near a special metric.

Keywords : Sub-Semi-Riemannian metrics, Ricci curvature, Einstein metrics, Inverse problem, Quasilinear elliptic systems.

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CONTENTS

1. INTRODUCTION

The goal of this note is to prove that the two principal results of D. DeTurck [11] given for positive definite symmetric bilinear form and for special Einstein metrics can be extended significantly in different ways.

Firstly, we can extend the positive definideness condition of the Riemannian metrics to Sub-Semi-Riemmannian metrics with the same rank and signature.

Secondly we are able to replace some particular Einstein metrics of non zero scalar curvature by any parallel Ricci metrics (ie. metrics with covariantly constant Ricci tensor).

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Let M be a smooth compact manifold without boundary. A Sub-Semi-Riemannian metric $\mathfrak{G}$ (SSR-metric for short) is a symmetric covariant 2-tensor field with constant signature and constant rank.

Let us now state the first result, interesting by itself, about almost conformal isometries.

Lemma 1.1. *Assume that $\mathfrak{G}$ and $\mathcal{G}$ are two smooth SSR-metrics on M with the same rank and signature. Let g be a smooth Riemmannian metrics on M, let $p \in [1, \infty)$ and let $\varepsilon > 0$. Then there exist a smooth diffemorphism Φ and a smooth positive function f such that $\Phi^*(f\mathcal{G}) - \mathfrak{G}$ is ε-close to zero in the L^p norm relative to g.*

Before going to the application for a Ricci equation, let us introduce some notations. For $(M, \mathfrak{g})$ a smooth riemannian manifold, we denote by $\mathrm{Ric}(\mathfrak{g})$ its Ricci curvature. For a real constant Λ, we consider the operator

$$\mathrm{Ein}(\mathfrak{g}) := \mathrm{Ric}(\mathfrak{g}) + \Lambda \mathfrak{g}.$$

This operator is geometric in the sense that for any smooth diffeomorphism φ,

$$\varphi^* \mathrm{Ein}(\mathfrak{g}) = \mathrm{Ein}(\varphi^* \mathfrak{g}).$$

We would like to invert Ein. We thus choose $\mathcal{E}$ a symmetric 2-tensor field on M, and look for $\mathfrak{g}$ Riemannian metric such that

$$\mathrm{Ein}(\mathfrak{g}) = \mathcal{E}. \tag{1.1}$$

This is a geometrically natural and difficult quasilinear system to solve, already for perturbation methods. The prescribed Ricci curvature problem has a long history starting with the work of D. DeTurck [9], [11], [10], [13], [12], [1], [14], [2], [3], [6], [8], [7], [5], [4],...

Motivated by the explosion of studies around the Ricci flow, and recently, some discrete versions thereof (eg. $\mathrm{Ein}(g_{i+1}) = g_i$), a renewed interest arises for this kind of natural geometric equations. We invite the reader to look at the nice recent works of A. Pulemotov and Y. Rubinstein [16] and [17] for related results. Our contribution here is the following.

Theorem 1.2. *Assume that M carries a Riemannian metric g with parallel Ricci tensor. Let $\Lambda \in \mathbb{R}$ such that $\mathrm{Ein}(g)$ is non degenerate, and that -2Λ is not in the spectrum of the Lichnerowicz Laplacian of g.*[1] *Then for any $\mathcal{E} \in C^\infty(M, \mathcal{S}_2)$ with the same rank and signature as $\mathrm{Ein}(g)$ at each point of M, there exist a smooth positive function f and a Riemannian metrics $\mathfrak{g}$ in $C^\infty(M, \mathcal{S}_2)$ such that*

$$\mathrm{Ein}(\mathfrak{g}) = f\,\mathcal{E}.$$

The proof goes by combining the Lemma 1.1 , the local inversion result of Proposition 3.1 for weak regular metric (where the conformal

[1] Like D. Deturck [11], we may allow an eigenspace spanned by g when $\Lambda = 0$

factor f is not required) and a regularity argument. We have then solved the problem up to a positive function f. Here we do not expect that f can be taken equals to one in general, this will be the subject of future investigations.

Parallel Ricci metrics, are (locally) products of Einstein metrics (see eg. [18]). They exists on the simplest examples of manifolds who do not admit Einstein metrics, like $\mathbb{S}^1 \times \mathbb{S}^2$, $\Sigma_g \times \mathbb{S}^2$, $(g \geq 1)$ or $\Sigma_g \times \mathbb{T}^2$, $(g \geq 2)$ where Σ_g is a surface of genius g. They are also static solutions of some geometric fourth order flows (eg. $\partial_t g = \Delta_g \operatorname{Ric}(g)$). Finally they are particular cases of Riemannian manifolds with Harmonic curvature (or equivalently Codazzi Ricci tensor).

Our global result show once again that such metrics with covariantly constant Ricci tensor deserve a particular attention.

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2. L^p closeness of some Sub-Semi-Riemannian metrics

We follow the section 3 called "approximation lemma" in [11] in order to verify that all the step there can be adapted for SSR-metrics as above. This will prove the Lemma 1.1.

We will keep almost the same notations as in [11], just replacing S and R there respectively by $\mathfrak{G}$ and $\mathcal{G}$ here.

Let $\mathfrak{G}$ and $\mathcal{G}$ be as in the introduction, we thus assume they have the same signature and the same rank. For the rest of the section we fix a Riemannian metric g, an $\varepsilon > 0$ and $p \in [1, +\infty)$. All the measures, volumes, and norms are understood with respect to g.

At each point $x \in M$, the two SSR-metric $\mathfrak{G}$ and $\mathcal{G}$ having the same rank and signature, there exists an orientation preserving automorphism u_x of T_xM, such that

$$\mathcal{G}_x(u_x(.), u_x(.)) = \mathfrak{G}_x(., .).$$

For $x \in M$, the following construction can be performed using the g-exponential map at x. There exists an open set U_x such that :

(i) U_x is contained in a coordinate neighborhood of x where in this coordinate (centered at 0), up to a positive automorphism u_x of T_xM, $\mathcal{G}$ is equal to $\mathfrak{G}$ at x:

$${}^t u_x \, \mathcal{G}_x \, u_x = \mathfrak{G}_x,$$

(ii) For any positive real α_x, the linear change of coordinates

$$\Phi_x := \sqrt{\alpha_x} \, u_x$$

satisfies on U_x the estimate (the left hand side of which does not depend upon α_x, and vanishes at the origin),

$$\left|(\Phi_x^* \frac{1}{\alpha_x} \mathcal{G})_y - \mathfrak{G}_y\right|^p \leq \min\left(\frac{\varepsilon^p}{2\mathrm{Vol}(M)}, |\mathfrak{G}_y|^p\right). \tag{2.1}$$

We consider a triangulation of M where each simplex S lies in the interior of some U_x with $x \in \mathring{S}$. Since the point x belong to the *interior* of the simplex S, shrinking α_x if necessary, we are sure that Φ_x send S into S (the norm of Φ_x approaches zero when α_x tends to zero).

Let Ω, Ω_1, Ω_2, Ω_3 be some open neighbourhoods of the $(n-1)$ dimensional skeleton (composed with union of the boundary of all simplex) with the properties :

$$\mathrm{Vol}(\Omega) < \frac{\varepsilon^p}{2(\max_M |\mathcal{G}| + 2\max_M |\mathfrak{G}|)^p},$$

and

$$\Omega_3 \subset \overline{\Omega}_3 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega.$$

The rest of the proof in section 3 of [11] is based on triangular inequalities between norms of tensors and can be implemented here without any change. For a better understanding, though, we provide further details of the figure page 368 of [11], specifying the estimates that occur on the differents parts of the simplex, see figure 1. On the picture, we have denoted the error $|\Phi^*(f\mathcal{G}) - \mathfrak{G}|$ by e :

$$e = |\Phi^*(f\mathcal{G}) - \mathfrak{G}|.$$

On the inner part T of the simplex, e is estimated by (2.1). The transition of the diffeomorphism Φ, on the middle ring $R_2 = S \cap (\Omega_1 \backslash \Omega_2)$, from Φ_x to the identity, still exist because our $\Phi_x = \sqrt{\alpha_x}\, u_x$ is an orientation preserving map with norm less than 1 as in [11].

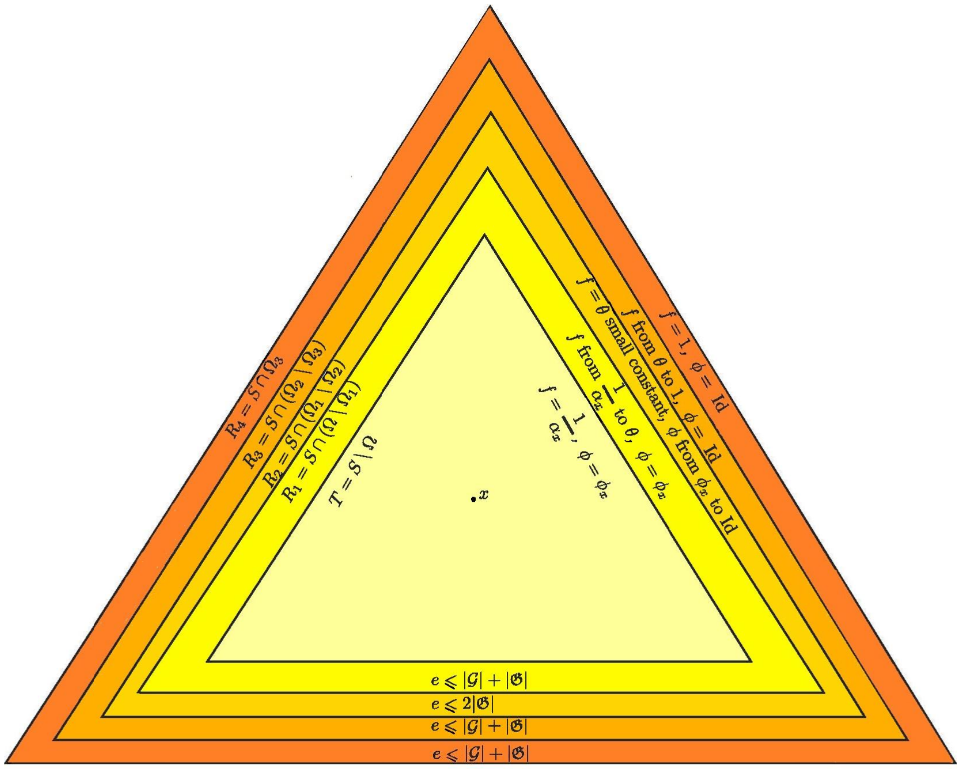

Figure 1. The simplex S with the values of f and Φ, and the estimates of e.

Exemple 2.1. The simplest non trivial example consists of a product of manifolds $M = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ with the two SSR-metrics of the form

$$\mathfrak{G}(x, y, z) = -g_{\mathcal{X},y,z}(x) \oplus g_{\mathcal{Y},x,z}(y) \oplus 0_{\mathcal{Z}},$$

where $g_{\mathcal{X},y,z}$ is a family of Riemannian metrics on $\mathcal{X}$, depending on the parameters y, z and smooth in all of its arguments. Here, any of the three manifolds but one may be reduced to a point.

3. Solvability of a Ricci type equation

We revisit the section 2 of [11] called "perturbation lemma".

We first need to introduce some operators. The divergence of a symmetric 2-tensor field and its L^2 adjoint acting on one form are

$$(\delta h)_j := -\nabla^i h_{ij} \ , \quad (\delta^* v)_{ij} := \frac{1}{2}(\nabla_i v_j + \nabla_j v_i).$$

The gravitationnal operator acting on symmetric 2-tensors is

$$G(h) := h - \frac{1}{2}\operatorname{Tr}_g(h)g.$$

The Lichnerowicz Laplacian is [2]

$$\Delta_L = \nabla^*\nabla + 2\operatorname{Ric} - 2\operatorname{Riem}.$$

It appears in the Linearization of the Ricci operator :

$$D\operatorname{Ric}(g) = \frac{1}{2}\Delta_L + \delta^*\delta\, G.$$

The Hodge Laplacian acting on one forms is

$$\Delta_H = \Delta + \operatorname{Ric} = \nabla^*\nabla + \operatorname{Ric} = d^*d + dd^*.$$

We also define the following Laplacian

$$\Delta_V := 2\,\delta\, G\,\delta^* = \nabla^*\nabla - \operatorname{Ric} = \Delta_H - 2\operatorname{Ric}.$$

We denote by V its finite dimensionnal kernel, composed of smooth one forms (by elliptic regularity).

We start with the equivalent of proposition 2.1 in [11].

Proposition 3.1. *Let (M, g) be a smooth Riemannian manifold with parallel Ricci curvature. Let $\Lambda \in \mathbb{R}$ such that $\operatorname{Ein}(g)$ is non degenerate and that -2Λ is not in the spectrum of the Lichnerowicz Laplacian. Let $k \in \mathbb{N}$ and $p > n$. Then for any $\mathcal{E}$ close to $\operatorname{Ein}(g)$ in $H^{k+1,p}(M, \mathcal{S}_2)$, there exist a Riemannian metrics $\mathfrak{g}$ in $H^{k+1,p}(M, \mathcal{S}_2)$ such that*

$$\operatorname{Ein}(\mathfrak{g}) = \mathcal{E}.$$

In [11], the proof of the corresponding proposition is given by a succession of lemmas. We thus revisit them one after the other. Some care is needed because we have to replace Ric and $-\Delta_L$ there, respectively with Ein and $\Delta_L + 2\Lambda$ here. Furthermore, in our context, the operator $\Delta_L + 2\Lambda$ has no kernel whereas the kernel of Δ_L is nonempty in [11], spanned by g. We clearly have also for any Riemannian metrics g, the Bianchi identity

$$\delta G(\operatorname{Ein}(g)) = 0.$$

We start with a local study of the action of the diffeomorphim group on the covariant symetrics 2-tensors, near a non degenerate parallel one. The result obtained remains in the spirit of the local study near a Riemmanian metric by Berger, Ebin, or Palais (see eg. the lemma 2.3 of [11]). Here the metric tensor is replaced by a non degenerate parallel tensor field.

[2] Different sign convention with DeTurck

Lemma 3.2. *Let E be a smooth, non degenerate and parallel symmetric two tensor field. Let $\mathcal{X}$ be a smooth Banach submanifold of $H^{k,p}(M,\mathcal{S}_2)$, whose tangent space at E is a complementary of $\delta^*(H^{k,p}(M,\mathcal{T}_1))$. Then for any $\mathcal{E}$ close enough to E in $H^{k,p}(M,\mathcal{S}_2)$, there exist an $H^{k+1,p}$ diffeomorphism Φ close to the identity such that $\Phi^*\mathcal{E} \in \mathcal{X}$.*

Proof. The tensor field E being parallel, its Lie derivative in the direction of a vector field v is

$$\mathcal{L}_v E = 2\delta^*(Ev).$$

Locally, the submanifold $\mathcal{X}$ can be seen as the image of an immersion $\mathfrak{X} : U \longrightarrow H^{k,p}(M,\mathcal{S}_2)$, with $\mathfrak{X}(0) = E$. We define $\mathcal{T}^\perp$ to be the set of vector fields $v \in H^{k+1,p}(M,\mathcal{T}_1)$ such that $E\,v$ is L^2-orthogonal [3] to $\ker\delta^*$.

Let

$$F : U \times \mathcal{T}^\perp \times H^{k,p}(M,\mathcal{S}_2) \longrightarrow H^{k,p}(M,\mathcal{S}_2),$$

defined by

$$F(k,Y,\mathcal{E}) = \Phi^*_{Y,1}(\mathcal{E}) - \mathfrak{X}(k),$$

where $\Phi_{Y,1}$ is the flow of the vector field Y at time 1. We have $F(0,0,E) = 0$ and the linearisation of F in the first two variables is

$$D_{(k,Y)}F(0,0,E)(l,X) = 2\delta^*(E\,X) - D\mathfrak{X}(0)l.$$

Now because

$$H^{k,p}(M,\mathcal{S}_2) = \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus \operatorname{Im} D\mathfrak{X}(0),$$

and E is non degenerate, then $D_{(k,Y)}F(0,0,E)$ is an isomorphism. From the implicit function theorem, for $\mathcal{E}$ close to E, there exist k and Y small such that $F(k,Y,\mathcal{E}) = 0$. $\square$

Let us recall the lemma 2.5 in [11] :[4]

Lemma 3.3. *For $k \geq 1$, we have*

$$H^{k,p}(M,\mathcal{S}_2) = \frac{(\ker\delta G \cap H^{k,p}(M,\mathcal{S}_2))}{\delta^*(V)} \oplus \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus G\delta^*(V).$$

The equivalent of the lemma 2.6 in [11] becomes (we do not have to quotient by $R\,g$ because of no kernel for us, and so we do not need to adjust with a constant c).

Lemma 3.4. *Suppose $k \geq 0$, $p > n$, and g satisfies the hypotheses of theorem 1.2 let*

$$K = \frac{(\ker\delta G \cap H^{k+2,p}(M,\mathcal{S}_2))}{\delta^*(V)} \oplus G\delta^*(V)$$

[3] closed complementary suffice

[4] It seems there is a misprint in the proof this lemma: The Ricci term for $\delta G\delta^*$ at the top of page 362 in [11] has a different sign.

and define $F : K \longrightarrow H^{k,p}(M,\mathcal{S}_2)$ *by*[5]

$$F(b) := \mathrm{Ein}(g+b).$$

Then for some neigbborood U *of* 0, $F(U)$ *is a Banach submanifold of* $H^{k,p}(M,\mathcal{S}_2)$ *whose tangent space at* $F(0)=\mathrm{Ein}(g)$ *is a complementary space of* $\delta^*(H^{k+1,p}(M,\mathcal{T}_1))$.

Proof. We have to show that the derivative of F at g is injective and its image has the closed subspace $\delta^*(H^{k+1,p})$ as complementary. A metric with parallel Ricci tensor is a local Einstein product so is smooth. We first show that the spaces $\mathrm{Im}\,\delta^*$ and $\ker\delta G$ are "stable" (modulo two points of regularity) by $\Delta_L+2\Lambda$ but also by $D\,\mathrm{Ein}(g)$ (when the metric is Ricci parallel). Indeed, recall that in that case [15]:

$$\delta\,\Delta_L = \Delta_H\,\delta,$$

and the adjoint version :

$$\Delta_L\,\delta^* = \delta^*\,\Delta_H.$$

We deduce that

$$(\Delta_L+2\Lambda)\,\delta^* = \delta^*\,(\Delta_H+2\Lambda) = \delta^*\,(\Delta_V+2\,\mathrm{Ein}),$$

and

$$D\,\mathrm{Ein}(g)\,\delta^* = \frac{1}{2}\delta^*\,(\Delta_H+2\Lambda)+\frac{1}{2}\delta^*\,\Delta_V = \delta^*\,(\Delta_V+\mathrm{Ein}) = \delta^*\,(\Delta+\Lambda)$$

thus the "stability" of $\mathrm{Im}\,\delta^*$ by the two operators above.

When restricted on the kernel of $\delta\,G$, we trivially have

$$D\,\mathrm{Ein}(g) = \frac{1}{2}(\Delta_L+2\Lambda),$$

but also, by linearising $\delta\,G\;\mathrm{Ein}(g)=0$ for instance,

$$\delta\,G\,(\Delta_L+2\Lambda)=0,$$

then the stability of $\ker\delta\,G$.

We can also remark that with the formula above, if $v\in V$,

$$(\Delta_L+2\Lambda)(\delta^*v) = 2\delta^*(\mathrm{Ein}\,v) = 2\,\mathrm{Ein}\,\delta^*v$$

and

$$D\,\mathrm{Ein}(g)(\delta^*v) = \delta^*(\mathrm{Ein}\,v) = \mathrm{Ein}\,\delta^*v.$$

For any function u, it is well known that $\Delta_L(ug)=(\Delta u)g$, so

$$(\Delta_L+2\Lambda)(d^*w\,g) = [(\Delta+2\Lambda)d^*w]g = [d^*(\Delta_H+2\Lambda)w]g.$$

We obtain that

$$(\Delta_L+2\Lambda)G\delta^* = G\delta^*(\Delta_H+2\Lambda).$$

If $v\in V$, we deduce

$$(\Delta_L+2\Lambda)G\delta^*v = G\delta^*(2\,\mathrm{Ein}\,v).$$

[5] To avoid ambiguities, we may take any fixed closed complementary W to $\delta^*(V)$ in $\ker\delta G\cap H^{k+2,p}(M,\mathcal{S}_2)$ instead of the first factor of K

Assume that -2Λ is not an eigenvalue of Δ_L, then $\Delta_L + 2\Lambda$ is an isomorphism from $H^{k+2,p}(M,\mathcal{S}_2)$ to $H^{k,p}(M,\mathcal{S}_2)$. The image of the splitting in Lemma 3.3 by $\Delta_L + 2\Lambda$ produce:[6]

$$H^{k,p}(M,\mathcal{S}_2) = \frac{(\ker \delta G \cap H^{k,p}(M,\mathcal{S}_2))}{\delta^*(\operatorname{Ein} V)} \oplus \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus G\delta^*(\operatorname{Ein} V).$$

The two first factors are the same than the image by $D\operatorname{Ein}(g)$ of the corresponding spaces in Lemma 3.3. Let us study the image of third one. For $v \in V$, we compute

$$\begin{aligned}\delta^*\delta GG\delta^* v &= \delta^*\delta G(\delta^* v + \frac{1}{2}d^*v\, g) = \frac{1}{2}\delta^*\delta G(d^*v\, g) = \frac{2-n}{4}\delta^*\delta(d^*v\, g)\\ &= \frac{n-2}{4}\delta^* dd^* v = \frac{n-2}{2}\delta^*\delta\delta^* v = -G\delta^*\delta\delta^* v. \end{aligned} \tag{3.1}$$

We deduce for instance that

$$D\operatorname{Ein}(g)G\delta^* V = \left[G\delta^*(\operatorname{Ein}\,.) + \frac{n-2}{2}\delta^*\delta\delta^*\right]V. \tag{3.2}$$

Let us define

$$\mathcal{F} := \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus G\delta^*(\operatorname{Ein} V).$$

We now prove that

$$\mathcal{F} = \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) \oplus D\operatorname{Ein}(g)G\delta^* V. \tag{3.3}$$

The fact that $\mathcal{F}$ is the sum of the two factors is clear by (3.2). Let w in the intersection of the two factors, so

$$w = \delta^* u = G\delta^*\operatorname{Ein} v + \delta^*\delta\delta^*\frac{n-2}{2}v,$$

for some $u \in H^{k+1,p}(M,\mathcal{T}_1)$ and $v \in V$. Because of the decomposition of $\mathcal{F}$, we deduce that $G\delta^*\operatorname{Ein} v = 0$ thus $(\Delta_L + 2\Lambda)G\delta^* v = 0$, then $G\delta^* v = 0$ and finally, by (3.1), $\frac{n-2}{2}\delta^*\delta\delta^* v = 0$ so $w = 0$. We have obtained

$$H^{k,p}(M,\mathcal{S}_2) = \operatorname{Im} DF(0) \oplus \delta^*(H^{k+1,p}(M,\mathcal{T}_1)).$$

We claim that $DF(0)$ is injective. Indeed, let h in the kernel of $DF(0)$, then $h = [u] + G\delta^* v$ with $[u]$ in the first summand of K. Thus $[\Delta_L + 2\Lambda][u] + D\operatorname{Ein}(g)G\delta^* v = 0$ so because of the decomposition (3.3), we obtain $[\Delta_L + 2\Lambda][u] = D\operatorname{Ein}(g)G\delta^* v = 0$. Its implies $[u] = 0$ and from equation (3.2), $v \in G\delta^*\operatorname{Ein} V \cap \delta^*(H^{k+1,p}(M,\mathcal{T}_1)) = \{0\}$, so $h = 0$. □

From the Lemma 3.2 with $E = \operatorname{Ein}(g)$ and Lemma 3.4 we directly deduce :

Lemma 3.5. *If $\mathcal{E} \in H^{k,p}$ and $|\mathcal{E} - \operatorname{Ein}(g)|_{k,p} < \varepsilon$, then there exist a metric $\mathfrak{g} \in H^{k+2,p}$ and a diffeomorphism $\varphi \in H^{k+1,p}$ for which $\operatorname{Ein}(\mathfrak{g}) = \varphi^*\mathcal{E}$.*

[6]Here also we have to replace the first factor by $(\Delta_L + 2\Lambda)W$ when a choice of W was made in the first factor of K.

We will complete the proof of proposition 3.1, where now $\mathcal{E} \in H^{k+1,p}$, but $\mathfrak{g}$ and φ still cames from Lemma 3.5 so φ is *a priori* not regular enough. If we inspect the pages 364-365 in [11] we can see that we just have to change $\mathrm{Ric}(\mathfrak{g})$ by $\mathrm{Ein}(\mathfrak{g})$ to obtain that φ is in fact in $H^{k+2,p}$. We conclude that $(\varphi^{-1})^*\mathfrak{g} \in H^{k+1,p}$ and has $\mathcal{E}$ as its image by Ein. At this level we also use that $\mathrm{Ein}(\mathfrak{g})$ is non degenerate (see equation (2.8) there).

The Theorem 1.2 is now a direct consequence of the Lemma 1.1, the Proposition 3.1 with $k = 0$, and the regularity result of [12].

Exemple 3.6. Recalling that the Ricci curvature of a product of Riemannian manifolds is the direct sum of the Ricci curvatures of each factors, we see that a product of Einstein manifolds clearly satisfies the assumption of the Theorem 1.2. The simplest example combining the 3 possibilities of Einstein constants is the following. Let us consider three compact Einstein manifolds $(\mathcal{X}, g_-)$, $(\mathcal{Y}, g_+)$, $(\mathcal{Z}, g_0)$ with Ricci curvatures given by $\mathrm{Ric}(g_-) = -g_-$, $\mathrm{Ric}(g_+) = g_+$, $\mathrm{Ric}(g_0) = 0$. Then $M = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ endowed with

$$g = g_- \oplus g_+ \oplus g_0,$$

has parallel Ricci curvature equal to

$$\mathrm{Ric}(g) = -g_- \oplus g_+ \oplus 0.$$

In this example, the kernel of Δ_L contains the parallel tensors

$$h = c_- g_- \oplus c_+ g_+ \oplus c_0 g_0,$$

for any constants c_-, c_+, c_0. Here, we only have to choose Λ in order to destroy this kernel and make $\mathrm{Ein}(g)$ non degenerate.

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