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# A notion of Denjoy sub-system

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## Abstract

We introduce a notion of Denjoy sub-system that generalizes the one of Aubry-Mather set. For such systems, we prove a result similar to Denjoy theorem (non-existence of  $C^2$  Denjoy sub-systems), and study their Lyapunov exponents. *To cite this article: M.-C. Arnaud, P. Le Calvez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## Résumé

**Une notion de sous-système de Denjoy.** Nous introduisons une notion de sous-système de Denjoy qui généralise celle d'ensemble d'Aubry-Mather. Pour ces systèmes, nous montrons un analogue du théorème de Denjoy (la non-existence de sous-systèmes de Denjoy de classe  $C^2$ ), et étudions leurs exposants de Lyapunov. *Pour citer cet article : M.-C. Arnaud, P. Le Calvez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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## Version française abrégée

**Remerciements.** Les auteurs remercient Raphael Krikorian de leur avoir posé la question de la régularité  $C^2$  des ensembles d'Aubry-Mather.

Nous introduisons une notion de sous-système de Denjoy pour un difféomorphisme d'une variété, notion qui généralise celle d'ensemble d'Aubry-Mather.

**Définition 0.1** *Soit  $f : M \rightarrow M$  un difféomorphisme de classe  $C^k$  d'une variété  $M$ . Un  $C^k$  (resp. Lipschitz) sous-système de Denjoy pour  $f$  est un triplet  $(K, \gamma, h)$  où*

—  $\gamma : \mathbb{T} \rightarrow M$  est un plongement de classe  $C^k$  (resp. biLipschitz);

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- $h : \mathbb{T} \rightarrow \mathbb{T}$  est un contre-exemple de Denjoy d'ensemble minimal  $K \subset \mathbb{T}$  ;
- $f(\gamma(K)) = \gamma(K)$  ;
- $\gamma \circ h|_K = f \circ \gamma|_K$ .

*Remarque 1* (i) Pour les difféomorphismes exact symplectiques de l'anneau qui dévient la verticale, la théorie d'Aubry-Mather fournit de nombreux sous-systèmes de Denjoy (voir [7]).

- (ii) D'autres ensembles de Cantor qui ne sont pas des sous-systèmes de Denjoy apparaissent de manière naturelle en dynamique. Par exemple, un fer à cheval a des points périodiques et un odomètre a des sous-ensembles fermés non triviaux périodiques.

Notre premier théorème énonce qu'une dynamique de Denjoy ne peut pas être insérée dans une courbe (non nécessairement invariante) de classe  $C^2$ .

**Théorème 0.2** *Il n'existe pas de sous-système de Denjoy de classe  $C^2$ .*

**Corollaire 0.3** *Soit  $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$  un difféomorphisme de classe  $C^2$  qui dévie la verticale (et n'est pas forcément symplectique). Si  $E \subset \mathbb{T} \times \mathbb{R}$  est un ensemble d'Aubry-Mather de nombre de rotation irrationnel contenu dans le graphe d'une fonction de classe  $C^2$   $\psi : \mathbb{T} \rightarrow \mathbb{R}$ , alors  $E$  coïncide avec ce graphe.*

L'entropie topologique et métrique d'un sous-système de Denjoy est nulle. Le résultat suivant exprime que l'unique mesure de probabilité définie par un sous-système de Denjoy de classe  $C^1$  a au moins un exposant de Lyapunov nul.

**Théorème 0.4** *Soit  $f$  un difféomorphisme de classe  $C^1$  d'une variété  $M$ . Si  $(K, \gamma, h)$  est un sous-système de Denjoy de classe  $C^1$  pour  $f$ , alors l'unique mesure de probabilité borélienne  $f$  invariante à support dans  $\gamma(K)$  a au moins un exposant de Lyapunov nul.*

**Corollaire 0.5** *Soit  $f$  un difféomorphisme de classe  $C^1$  qui préserve l'aire d'une surface  $M$ . Si  $(K, \gamma, h)$  est un sous-système de Denjoy de classe  $C^1$  pour  $f$ , alors l'unique mesure de probabilité borélienne  $f$  invariante à support dans  $\gamma(K)$  a ses deux exposants de Lyapunov nuls.*

*Remarque 2* (i) Le second auteur a montré dans [6] qu'un difféomorphisme générique symplectique déviant la verticale a beaucoup d'ensembles d'Aubry-Mather qui sont uniformément hyperboliques. Parmi ceux-ci, ceux qui ont un nombre de rotation irrationnel définissent alors des sous-systèmes de Denjoy qui sont Lipschitz mais pas de classe  $C^1$ .

- (ii) Dans [2], il est montré que l'unique mesure de probabilité invariante portée par un lacet invariant d'un difféomorphisme de classe  $C^{1+\alpha}$  ne contenant pas de point périodique a au moins un exposant de Lyapunov nul. La première remarque montre que ce résultat ne peut pas être étendu aux sous-systèmes de Denjoy Lipschitz définis à l'aide d'une courbe non invariante.

- (iii) Le premier auteur a montré dans [1] une sorte de réciproque au corollaire 0.5 : si les exposants de Lyapunov de la mesure à support dans un ensemble d'Aubry-Mather de nombre de rotation irrationnel d'une application symplectique déviant la verticale sont nuls, alors le support de la mesure est en un certain sens de classe  $C^1$  presque partout. La proposition suivante montre que ce résultat est faux si on enlève l'hypothèse de déviation de la verticale.

**Proposition 0.6** *Il existe un difféomorphisme de classe  $C^1$   $F$  de  $\mathbb{T} \times \mathbb{R}$  qui admet un sous-système Lipschitz de Denjoy  $(K, \gamma, h)$  tel que*

- les exposants de Lyapunov de l'unique mesure de probabilité à support dans  $\gamma(\mathbb{T})$  sont nuls ;
- il n'existe pas de sous-système de Denjoy de classe  $C^1$   $(K_0, \gamma_0, h_0)$  pour un difféomorphisme de classe  $C^1$   $f$  de  $\mathbb{T} \times \mathbb{R}$  tel que  $\gamma_0(K_0) = \gamma(K)$ .

*Remarque 3* L'exemple de sous-système de Denjoy construit dans la proposition 0.6 admet en fait un feuilletage Lipschitz de courbes invariantes dont chacune porte un contre-exemple de Denjoy.

Cette notion de sous-système de Denjoy s'étend pour les flots.

**Définition 0.7** Soit  $(\varphi_t)_{t \in \mathbb{R}}$  un flot de classe  $C^k$  d'une variété  $M$ . Un sous-flot de Denjoy de classe  $C^k$  (resp. Lipschitz) pour  $(\varphi_t)_{t \in \mathbb{R}}$  est un triplet  $(K, j, (\psi_t)_{t \in \mathbb{R}})$  où

- $j : \mathbb{T}^2 \rightarrow M$  est un plongement de classe  $C^k$  (resp. Lipschitz);
- $(\psi_t)_{t \in \mathbb{R}}$  est un flot de classe  $C^1$  sur  $\mathbb{T}^2$  sans point périodique ni orbite dense;
- si  $K$  est l'unique ensemble fermé minimal invariant de  $(\psi_t)_{t \in \mathbb{R}}$ , alors  $j(K)$  est invariant par  $(\varphi_t)_{t \in \mathbb{R}}$ ;
- pour tout  $x \in K$  et tout  $t \in \mathbb{R}$ , on a  $j \circ \psi_t(x) = \varphi_t \circ j(x)$ .

**Théorème 0.8** Il n'existe pas de sous-flot de Denjoy de classe  $C^2$ .

Un résultat similaire pour les graphes est prouvé dans [4] dans le cadre de la théorie K.A.M. faible.

## 1. Introduction

In 1932, Arnaud Denjoy proved in [3] that if a  $C^2$  vector-field of  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  has a rotation vector that does not belong to  $\mathbb{R}\mathbb{Z}^2$ , then every orbit is dense in  $\mathbb{T}^2$ . A similar result for diffeomorphisms of  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  can be stated in the following way: if an orientation preserving  $C^2$  diffeomorphism of the circle has an irrational rotation number, then it is  $C^0$  conjugated to a rotation. There exist examples of orientation preserving  $C^1$  diffeomorphisms of  $\mathbb{T}$  that have an irrational rotation number and are not  $C^0$  conjugated to a rotation, or equivalently are not minimal (see [5] for example). Such examples in any regularity (i.e.  $C^0$  or  $C^1$ ) are now called *Denjoy counter-examples*, and they have a unique minimal invariant closed set that is a Cantor set (i.e. compact, totally disconnected and with no isolated point).

This implies of course that if a  $C^2$  diffeomorphism  $f$  of a manifold has an invariant loop  $\Gamma$  such that  $f|_\Gamma$  has an irrational rotation number and is not minimal, then  $\Gamma$  is not  $C^2$ . Here we address the question when we don't require the curve to be invariant: can a Denjoy dynamics be contained in some non invariant  $C^2$  curve? In other words, can the set that supports this dynamics be  $C^2$  in some sense? To explain in detail this question, we have to introduce some notions.

**Definition 1.1** Let  $f : M \rightarrow M$  be a  $C^k$  diffeomorphism of a manifold  $M$ . A  $C^k$  (resp. Lipschitz) Denjoy sub-system for  $f$  is a triplet  $(K, \gamma, h)$  where

- $\gamma : \mathbb{T} \rightarrow M$  is a  $C^k$  (resp. biLipschitz) embedding;
- $h : \mathbb{T} \rightarrow \mathbb{T}$  is a Denjoy counter-example with invariant minimal set  $K \subset \mathbb{T}$ ;
- $f(\gamma(K)) = \gamma(K)$ ;
- $\gamma \circ h|_K = f \circ \gamma|_K$ .

*Remark 1* (i) For area preserving twist maps of the annulus, Aubry-Mather theory tells us that there are many Lipschitz Denjoy sub-systems (see [7]).

(ii) Other kinds of invariant Cantor sets appear naturally in dynamical systems that are not Denjoy sub-systems. For example a horseshoe has periodic points, an odometer has periodic non trivial closed subsets.

Our first result asserts that a Denjoy sub-system is never  $C^2$ .

**Theorem 1.2** There exist no  $C^2$  Denjoy sub-system.

We immediately get:

**Corollary 1.3** *Let  $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$  be a  $C^2$  twist map (non necessarily area preserving). If  $E \subset \mathbb{T} \times \mathbb{R}$  is an Aubry-Mather set of irrational rotation number that is contained in the graph of a  $C^2$  function  $\psi : \mathbb{T} \rightarrow \mathbb{R}$ , then  $E$  coincides with this graph.*

The topological entropy of a circle homeomorphism is equal to zero. Moreover such a homeomorphism is uniquely ergodic in case its rotation number is irrational. Consequently the restriction of a diffeomorphism of a manifold to a Denjoy sub-system has zero entropy and is uniquely ergodic. The next result asserts that the unique invariant probability measure has one zero Lyapunov in case the sub-system is  $C^1$ .

**Theorem 1.4** *Let  $f$  be a  $C^1$  diffeomorphism of a manifold  $M$ . If  $(K, \gamma, h)$  is a  $C^1$  Denjoy sub-system for  $f$ , then the unique  $f$  invariant Borel probability measure that is supported on  $\gamma(K)$  has at least one vanishing Lyapunov exponent.*

We easily deduce:

**Corollary 1.5** *Let  $f$  be an area preserving  $C^1$  diffeomorphism of a surface  $M$ . If  $(K, \gamma, h)$  is a  $C^1$  Denjoy sub-system for  $f$ , then the unique  $f$  invariant Borel probability measure supported on  $\gamma(K)$  has its two Lyapunov exponents equal to zero.*

*Remark 2 (i) The second author proved in [6] that for a generic area preserving twist map, many Aubry-Mather sets are uniformly hyperbolic. Such an Aubry-Mather set cannot be a loop. Among them, those that have an irrational rotation number are Lipschitz Denjoy sub-systems but not  $C^1$  Denjoy sub-systems.*

*(ii) It was proved in [2] that any continuous invariant loop by a  $C^{1+\alpha}$  diffeomorphism of a surface that contains no periodic point carries a unique invariant measure that has at least one zero Lyapunov exponent. The first remark shows that this result cannot be extended to Lipschitz Denjoy sub-systems that are not defined via an invariant curve.*

*(iii) The first author proved in [1] a kind of reverse result of Corollary 1.5: if the Lyapunov exponents of a measure that is supported on an Aubry-Mather set with irrational rotation number of a symplectic twist map are zero, then the support of the measure is, in some sense,  $C^1$  regular almost everywhere. The next proposition gives a counter-example in the non twist setting.*

**Proposition 1.6** *There exists a  $C^1$  diffeomorphism  $F$  of  $\mathbb{T} \times \mathbb{R}$  that admits a Lipschitz Denjoy sub-system  $(K, \gamma, h)$  such that*

- *the Lyapunov exponents of the unique invariant probability measure that is supported on  $\gamma(\mathbb{T})$  are equal to zero;*
- *there exists no  $C^1$  Denjoy sub-system  $(K_0, \gamma_0, h_0)$  for a  $C^1$  diffeomorphism  $f$  of  $\mathbb{T} \times \mathbb{R}$  such that  $\gamma_0(K_0) = \gamma(K)$ .*

*Remark 3 The diffeomorphism built in Proposition 1.6 has a Lipschitz foliation into invariant Lipschitz graphs on which the dynamics is a Denjoy counter-example.*

This notion of Denjoy sub-system can be extended to flows.

**Definition 1.7** *Let  $(\varphi_t)_{t \in \mathbb{R}}$  be a  $C^k$  flow on a manifold  $M$ . A  $C^k$  (resp. Lipschitz) Denjoy sub-flow for  $(\varphi_t)_{t \in \mathbb{R}}$  is a triplet  $(K, j, (\psi_t)_{t \in \mathbb{R}})$  where*

- *$j : \mathbb{T}^2 \rightarrow M$  is a  $C^k$  (resp. Lipschitz) embedding;*
- *$(\psi_t)_{t \in \mathbb{R}}$  is a  $C^1$  flow of  $\mathbb{T}^2$  with no periodic orbit nor dense orbit;*
- *if  $K$  is the unique minimal invariant closed subset of  $(\psi_t)_{t \in \mathbb{R}}$ , then  $j(K)$  is invariant by  $(\varphi_t)_{t \in \mathbb{R}}$ ;*
- *for every  $x \in K$  and every  $t \in \mathbb{R}$ , one has  $j \circ \psi_t(x) = \varphi_t \circ j(x)$ .*

**Theorem 1.8** *There exists no  $C^2$  Denjoy sub-flow.*

In [4], a similar result for graphs is proved in the setting of weak K.A.M. theory.

## 2. Proof of Theorems 1.2 and 1.4

The key result, in the proofs of Theorems 1.2 et 1.4 is the following:

**Proposition 2.1** *Let  $k \geq 1$  be some integer. Let  $(K, \gamma, h)$  be a  $C^k$  Denjoy sub-system for some  $C^k$ -diffeomorphism  $f : M \rightarrow M$ . There exists an orientation preserving  $C^k$  diffeomorphism  $\varphi : \mathbb{T} \rightarrow \mathbb{T}$  such that  $\varphi|_K = h|_K$ .*

*Proof.* With the notations of Proposition 2.1, observe that the two loops  $\gamma : \mathbb{T} \rightarrow M$  and  $f \circ \gamma : \mathbb{T} \rightarrow M$  are tangent along the  $f$  invariant Cantor set  $\gamma(K)$ . Indeed, this set is included in both loops and has no isolated point.

Let  $\mathcal{N}$  be a tubular neighborhood of  $\gamma(\mathbb{T})$  in which we can define a  $C^k$  projection  $p : \mathcal{N} \rightarrow \gamma(\mathbb{T})$ . Observe that  $p \circ f \circ \gamma$ , coinciding with  $\gamma \circ h$  on  $K$ , is a  $C^k$  embedding when restricted to a neighborhood of  $K$ . This implies that there exist two open neighborhoods  $U, U'$  of  $K$  in  $\mathbb{T}$  such that  $f \circ \gamma(U) \subset \mathcal{N}$  and such that  $\gamma^{-1} \circ p \circ f \circ \gamma$  induces a  $C^k$  diffeomorphism between  $U$  and  $U'$ . The family of connected components of  $U$  is an open covering of  $K$ . By compactness of  $K$ , there are finitely many componeets that meet  $K$ . So, taking a subset of  $U$  if necessary, we can assume that the number of connected components of  $U$  is finite, larger than 1, and that each connected component meets  $K$ . We can index these components, getting a family  $(U_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  ordered in the usual cyclic order along  $\mathbb{T}$ . Setting  $U'_i = \gamma^{-1} \circ p \circ f \circ \gamma(U_i)$ , one gets another family  $(U'_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  ordered in the usual cyclic order along  $\mathbb{T}$  because  $\gamma^{-1} \circ p \circ f \circ \gamma|_K = h|_K$ . We can find a covering  $(V_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$  of  $K$  by open and connected subsets of  $\mathbb{T}$ , satisfying  $\bar{V}_i \subset U_i$  for every  $i \in \mathbb{Z}/n\mathbb{Z}$ . For every  $j \in \mathbb{Z}/n\mathbb{Z}$ , we define  $V'_j = \gamma^{-1} \circ p \circ f \circ \gamma(V_j)$  and we denote  $J_i$  the connected component of the complement of  $\bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} V_i$  that lies between  $V_i$  and  $V_{i+1}$  and  $J'_i$  the connected component of the

complement of  $\bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} V'_i$  that lies between  $V'_i$  and  $V'_{i+1}$ . To extend  $f|_{\bigcup_{1 \leq i \leq n} \bar{V}_i}$  to a  $C^k$  diffeomorphism of

$\mathbb{T}$ , it is sufficient to find, for every  $i \in \mathbb{Z}/n\mathbb{Z}$ , an increasing  $C^k$  diffeomorphism  $\varphi_i : J_i \rightarrow J'_i$  with prescribed values of the derivatives  $\varphi_i^{(l)}$ ,  $1 \leq l \leq k$ , at the two ends of  $J_i$  to ensure a global differentiability. There is no obstruction to such a construction.  $\square$

Denjoy Theorem implies that under the assumption of Proposition 2.1, the integer  $k$  must be equal to 1, so Theorem 1.2 is proved. Moreover, when  $k = 1$ , then  $\mathbb{R}.\gamma'$  is a  $Df$  invariant lines bundle along  $\gamma(K)$  that satisfies  $Df.\gamma' = \varphi'(\gamma' \circ \varphi)$ . So, for every integer  $n \geq 0$  one has

$$Df^n.\gamma' = (\varphi^n)'(\gamma' \circ \varphi^n).$$

We deduce that the unique  $f$  invariant probability measure supported on  $\gamma(K)$  has a Lyapunov exponent that coincides with the Lyapounov exponent of the unique invariant measure of  $\varphi$ , a  $C^1$  diffeomorphism of  $\mathbb{T}$ . This Lyapunov exponent being known to vanish, Theorem 1.4 is proved.

## 3. Proof of Proposition 1.6

As explained in [5], one can construct a  $C^1$  Denjoy counter-example  $g : \mathbb{T} \rightarrow \mathbb{T}$  such that:

- $g$  has exactly two orbits of wandering intervals  $(I_k)_{k \in \mathbb{Z}}$  and  $(J_k)_{k \in \mathbb{Z}}$  where  $\text{Leb}(I_k) = \text{Leb}(J_k) = \ell_k$ ;
- $\sum_{k \in \mathbb{Z}} \ell_k = \frac{1}{2}$ ;
- $\lim_{k \rightarrow \pm\infty} \frac{\ell_k}{\ell_{k+1}} = 1$ ;
- the derivative  $g'$  is equal to 1 on the minimal compact invariant set  $K$ .

In particular, one has  $\mathbb{T} \setminus K = \bigcup_{k \in \mathbb{Z}} \text{Int}(I_k) \cup \bigcup_{k \in \mathbb{Z}} \text{Int}(J_k)$ .

There exists a Lipschitz function  $\psi : \mathbb{T} \rightarrow \mathbb{R}$ , uniquely defined up to an additive constant, such that

$$\psi' \Big|_{\bigcup_{k \in \mathbb{Z}} \text{Int}(I_k)} = 1, \quad \psi' \Big|_{\bigcup_{k \in \mathbb{Z}} \text{Int}(J_k)} = -1.$$

Observe that  $\psi'$  is constant along every orbit of  $g$  that is contained in  $\mathbb{T} \setminus K$ .

The map  $\eta : \mathbb{T} \rightarrow \mathbb{R}$  defined by  $\eta = \psi \circ g - \psi$  is also Lipschitz and differentiable on  $\mathbb{T} \setminus K$ . Moreover, for every  $\theta \in \mathbb{T} \setminus K$ , one has

$$\eta'(\theta) = g'(\theta) \psi'(g(\theta)) - \psi'(\theta) = \psi'(\theta) (g'(\theta) - 1).$$

The fact that  $\psi'$  is bounded on  $\mathbb{T} \setminus K$  and that  $1 - g'$  is continuous on  $\mathbb{T}$  and vanishes on  $K$  implies that  $\eta$  is  $C^1$ .

Finally, let us define a  $C^1$  diffeomorphism  $F$  of  $\mathbb{T} \times \mathbb{R}$  by the formula

$$F(\theta, r) = (g(\theta), r + \eta(\theta)).$$

The graph of  $\psi$  is invariant by  $F$  because

$$F(\theta, \psi(\theta)) = (g(\theta), \psi(\theta) + \eta(\theta)) = (g(\theta), \psi \circ g(\theta)),$$

Note that for every  $\theta \in \mathbb{T}$  and every  $k \in \mathbb{Z}$ , one has

$$DF^k(\theta_0, \psi(\theta_0)) = \begin{pmatrix} (g^k)'(\theta_0) & 0 \\ b_k & 1 \end{pmatrix}$$

Hence the two Lyapunov exponents of the unique invariant measure that is supported on the graph of  $\psi$  are zero.

Moreover, if  $\theta \in K$ , there exists two sequences  $(I_{i_k})_{k \in \mathbb{N}}$  and  $(J_{j_k})_{k \in \mathbb{N}}$  that converge to  $\theta$ . The slope of  $\psi$  between the two ends of  $I_{i_k}$  is 1 and the slope of  $\psi$  between the two ends of  $J_{j_k}$  is -1. This implies that there exists no  $C^1$  Denjoy sub-system  $(K_0, \gamma_0, h_0)$  of a diffeomorphism  $f$  of  $\mathbb{T} \times \mathbb{R}$  such that  $\gamma_0(K_0) = \text{graph}(\psi|_K)$ .

#### 4. Proof of Theorem 1.8

We assume that  $(K_0, j, (\psi_t)_{t \in \mathbb{R}})$  is a  $C^2$  Denjoy sub-flow for some  $C^2$  flow  $(\varphi_t)_{t \in \mathbb{R}}$  on a manifold  $M$ . We choose on  $\mathbb{T}^2$  an essential  $C^2$  embedding  $\gamma : \mathbb{T} \rightarrow \mathbb{T}^2$  that is transverse to the flow direction  $\dot{\psi}$  of  $(\psi_t)_{t \in \mathbb{R}}$ . The first return map  $r : \gamma(\mathbb{T}) \rightarrow \gamma(\mathbb{T})$  is then well defined, of class  $C^1$ , and the map  $h = \gamma^{-1} \circ r \circ \gamma : \mathbb{T} \rightarrow \mathbb{T}$  is a  $C^1$  Denjoy counter-example. We denote  $K$  the minimal set of  $h$  and set  $\mathcal{K} = j \circ \gamma(K)$  and  $\mathcal{T} = j \circ \gamma(\mathbb{T})$ .

Let  $\mathcal{N}$  be a tubular neighborhood of  $j(\mathbb{T}^2)$  in which we can define a  $C^2$  projection  $p : \mathcal{N} \rightarrow j(\mathbb{T}^2)$  and consider the  $C^2$  hypersurface  $\mathcal{H} = p^{-1}(\mathcal{T})$ . The flow direction  $\dot{\varphi}$  of  $(\varphi_t)_{t \in \mathbb{R}}$  is transverse to  $\mathcal{H}$  on  $\mathcal{K}$ . Moreover the first return map of  $(\varphi_t)_{t \in \mathbb{R}}$  on  $\mathcal{H}$  is well defined on  $\mathcal{K}$  and coincide with  $j \circ \gamma \circ h \circ \gamma^{-1} \circ j^{-1}|_{\mathcal{K}}$ . One deduces that there exist two neighborhoods  $\mathcal{W}$  and  $\mathcal{W}'$  of  $\mathcal{K}$  in  $\mathcal{H}$  such that the first return map of

$(\varphi_t)_{t \in \mathbb{R}}$  on  $\mathcal{H}$  is well defined on  $\mathcal{W}$  and induces a  $C^2$  diffeomorphism  $\mathcal{R}$  between  $\mathcal{W}$  and  $\mathcal{W}'$ . The map  $\gamma^{-1} \circ j^{-1}|_{\mathcal{T}} \circ p \circ \mathcal{R} \circ j \circ \gamma$  is a  $C^2$  map defined on a neighborhood of  $K$  with values in  $\mathbb{T}$  that coincides with the  $C^1$  diffeomorphism  $h$  on  $K$ . So it induces a  $C^2$  diffeomorphism between two neighborhoods  $U$ ,  $U'$  of  $K$ . As in the proof of Proposition 2.1, we may require that  $U$  and  $U'$  have finitely many connected components and we can construct a  $C^2$  diffeomorphism of  $\mathbb{T}$  that coincides with  $h$  in a neighborhood of  $K$ . This gives a contradiction.

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