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Dynamical systems

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**Abstract**

We introduce a notion of Denjoy sub-system that generalizes the one of Aubry-Mather set. For such systems, we prove a result similar to Denjoy theorem (non-existence of $C^2$ Denjoy sub-systems), and study their Lyapunov exponents. *To cite this article: M.-C. Arnaud, P. Le Calvez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

**Résumé**

**Une notion de sous-système de Denjoy.** Nous introduisons une notion de sous-système de Denjoy qui généralise celle d'ensemble d'Aubry-Mather. Pour ces systèmes, nous montrons un analogue du théorème de Denjoy (la non-existence de sous-systèmes de Denjoy de classe $C^2$), et étudions leurs exposants de Lyapunov. *Pour citer cet article : M.-C. Arnaud, P. Le Calvez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## Version française abrégée

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Nous introduisons une notion de sous-système de Denjoy pour un difféomorphisme d'une variété, notion qui généralise celle d'ensemble d'Aubry-Mather.

**Définition** 0.1 *Soit $f : M \to M$ un difféomorphisme de classe $C^k$ d'une variété $M$. Un $C^k$* (resp. Lipschitz) sous-système de Denjoy pour *$f$ est un triplet $(K, \gamma, h)$ où*

— $\gamma : \mathbb{T} \to M$ *est un plongement de classe $C^k$ (resp. biLipschitz) ;*

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— *$h : \mathbb{T} \to \mathbb{T}$ est un contre-exemple de Denjoy d'ensemble minimal $K \subset \mathbb{T}$ ;*
— *$f(\gamma(K)) = \gamma(K)$ ;*
— *$\gamma \circ h_{|K} = f \circ \gamma_{|K}$.*

*Remarque 1* *(i) Pour les difféomorphismes exact symplectiques de l'anneau qui dévient la verticale, la théorie d'Aubry-Mather fournit de nombreux sous-systèmes de Denjoy (voir [7]).*

*(ii) D'autres ensembles de Cantor qui ne sont pas des sous-systèmes de Denjoy apparaissent de manière naturelle en dynamique. Par exemple, un fer à cheval a des points périodiques et un odomètre a des sous-ensembles fermés non triviaux périodiques.*

Notre premier théorème énonce qu'une dynamique de Denjoy ne peut pas être insérée dans une courbe (non nécessairement invariante) de classe $C^2$.

**Théorème 0.2** *Il n'existe pas de sous-système de Denjoy de classe $C^2$.*

**Corollaire 0.3** *Soit $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ un difféomorphisme de classe $C^2$ qui dévie la verticale (et n'est pas forcément symplectique). Si $E \subset \mathbb{T} \times \mathbb{R}$ est un ensemble d'Aubry-Mather de nombre de rotation irrationnel contenu dans le graphe d'une fonction de classe $C^2$ $\psi : \mathbb{T} \to \mathbb{R}$, alors $E$ coïncide avec ce graphe.*

L'entropie topologique et métrique d'un sous-système de Denjoy est nulle. Le résultat suivant exprime que l'unique mesure de probabilité définie par un sous-système de Denjoy de classe $C^1$ a au moins un exposant de Lyapunov nul.

**Théorème 0.4** *Soit $f$ un difféomorphisme de classe $C^1$ d'une variété $M$. Si $(K, \gamma, h)$ est un sous-système de Denjoy de classe $C^1$ pour $f$, alors l'unique mesure de probabilité borélienne $f$ invariante à support dans $\gamma(K)$ a au moins un exposant de Lyapunov nul.*

**Corollaire 0.5** *Soit $f$ un difféomorphisme de classe $C^1$ qui préserve l'aire d'une surface $M$. Si $(K, \gamma, h)$ est un sous-système de Denjoy de classe $C^1$ pour $f$, alors l'unique mesure de probablité borélienne $f$ invariante à suppport dans $\gamma(K)$ a ses deux exposants de Lyapunov nuls.*

*Remarque 2* *(i) Le second auteur a montré dans [6] qu'un difféomorphisme générique symplectique déviant la verticale a beaucoup d'ensembles d'Aubry-Mather qui sont uniformément hyperboliques. Parmi ceux-ci, ceux qui ont un nombre de rotation irrationnel définissent alors des sous-systèmes de Denjoy qui sont Lipschitz mais pas de classe $C^1$.*

*(ii) Dans [2], il est montré que l'unique mesure de probabilité invariante portée par un lacet invariant d'un difféomorphisme de classe $C^{1+\alpha}$ ne contenant pas de point périodique a au moins un exposant de Lyapunov nul. La première remarque montre que ce résultat ne peut pas être étendu aux sous-systèmes de Denjoy Lipschitz définis à l'aide d'une courbe non invariante.*

*(iii) Le premier auteur a montré dans [1] une sorte de réciproque au corollaire 0.5 : si les exposants de Lyapunov de la mesure à support dans un ensemble d'Aubry-Mather de nombre de rotation irrationnel d'une application symplectique déviant la verticale sont nuls, alors le support de la mesure est en un certain sens de classe $C^1$ presque partout. La proposition suivante montre que ce résultat est faux si on enlève l'hypothèse de déviation de la verticale.*

**Proposition 0.6** *Il existe un difféomorphisme de classe $C^1$ $F$ de $\mathbb{T} \times \mathbb{R}$ qui admet un sous-système Lipschitz de Denjoy $(K, \gamma, h)$ tel que*
— *les exposants de Lyapunov de l'unique mesure de probabilité à support dans $\gamma(\mathbb{T})$ sont nuls ;*
— *il n'existe pas de sous-système de Denjoy de classe $C^1$ $(K_0, \gamma_0, h_0)$ pour un difféomorphisme de classe $C^1$ $f$ de $\mathbb{T} \times \mathbb{R}$ tel que $\gamma_0(K_0) = \gamma(K)$.*

*Remarque 3 L'exemple de sous-système de Denjoy construit dans la proposition 0.6 admet en fait un feuilletage Lipschitz de courbes invariantes dont chacune porte un contre-exemple de Denjoy.*

Cette notion de sous-système de Denjoy s'étend pour les flots.

**Définition** 0.7 *Soit $(\varphi_t)_{t\in\mathbb{R}}$ un flot de classe $C^k$ d'une variété $M$. Un* sous-flot de Denjoy de classe $C^k$ (resp. Lipschitz) pour $(\varphi_t)_{t\in\mathbb{R}}$ *est un triplet $(K, j, (\psi_t)_{t\in\mathbb{R}})$ où*

— *$j : \mathbb{T}^2 \to M$ est un plongement de classe $C^k$ (resp. Lipschitz) ;*
— *$(\psi_t)_{t\in\mathbb{R}}$ est un flot de classe $C^1$ sur $\mathbb{T}^2$ sans point périodique ni orbite dense ;*
— *si $K$ est l'unique ensemble fermé minimal invariant de $(\psi_t)_{t\in\mathbb{R}}$, alors $j(K)$ est invariant par $(\varphi_t)_{t\in\mathbb{R}}$ ;*
— *pour tout $x \in K$ et tout $t \in \mathbb{R}$, on a $j \circ \psi_t(x) = \varphi_t \circ j(x)$.*

**Théorème 0.8** *Il n'existe pas de sous-flot de Denjoy de classe $C^2$.*

Un résultat similaire pour les graphes est prouvé dans [4] dans le cadre de la théorie K.A.M. faible.

## 1. Introduction

In 1932, Arnaud Denjoy proved in [3] that if a $C^2$ vector-field of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ has a rotation vector that does not belong to $\mathbb{R}.\mathbb{Z}^2$, then every orbit is dense in $\mathbb{T}^2$. A similar result for diffeomorphisms of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be stated in the following way: if an orientation preserving $C^2$ diffeomorphism of the circle has an irrational rotation number, then it is $C^0$ conjugated to a rotation. There exist examples of orientation preserving $C^1$ diffeomorphisms of $\mathbb{T}$ that have an irrational rotation number and are not $C^0$ conjugated to a rotation, or equivalently are not minimal (see [5] for example). Such examples in any regularity (i.e. $C^0$ or $C^1$) are now called *Denjoy counter-examples*, and they have a unique minimal invariant closed set that is a Cantor set (i.e. compact, totally disconnected and with no isolated point).

This implies of course that if a $C^2$ diffeomorphism $f$ of a manifold has an invariant loop $\Gamma$ such that $f_{|\Gamma}$ has an irrational rotation number and is not minimal, then $\Gamma$ is not $C^2$. Here we address the question when we don't require the curve to be invariant: can a Denjoy dynamics be contained in some non invariant $C^2$ curve? In other words, can the set that supports this dynamics be $C^2$ in some sense? To explain in detail this question, we have to introduce some notions.

**Definition** 1.1 *Let $f : M \to M$ be a $C^k$ diffeomorphism of a manifold $M$. A $C^k$* (resp. Lipschitz) Denjoy sub-system for *$f$ is a triplet $(K, \gamma, h)$ where*

— *$\gamma : \mathbb{T} \to M$ is a $C^k$ (resp. biLipschitz) embedding;*
— *$h : \mathbb{T} \to \mathbb{T}$ is a Denjoy counter-example with invariant minimal set $K \subset \mathbb{T}$;*
— *$f(\gamma(K)) = \gamma(K)$;*
— *$\gamma \circ h_{|K} = f \circ \gamma_{|K}$.*

*Remark 1* (i) *For area preserving twist maps of the annulus, Aubry-Mather theory tells us that there are many Lipschitz Denjoy sub-systems (see [7]).*

(ii) *Other kinds of invariant Cantor sets appear naturally in dynamical systems that are not Denjoy sub-systems. For example a horseshoe has periodic points, an odometer has periodic non trivial closed subsets.*

Our first result asserts that a Denjoy sub-system is never $C^2$.

**Theorem 1.2** *There exist no $C^2$ Denjoy sub-system.*

We immediately get:

**Corollary 1.3** *Let $f : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ be a $C^2$ twist map (non necessarily area preserving). If $E \subset \mathbb{T}\times\mathbb{R}$ is an Aubry-Mather set of irrational rotation number that is contained in the graph of a $C^2$ function $\psi : \mathbb{T} \to \mathbb{R}$, then $E$ coincides with this graph.*

The topological entropy of a circle homeomorphism is equal to zero. Moreover such a homeomorphism is uniquely ergodic in case its rotation number is irrational. Consequently the restriction of a diffeomorphism of a manifold to a Denjoy sub-system has zero entropy and is uniquely ergodic. The next result asserts that the unique invariant probability measure has one zero Lyapunov in case the sub-system is $C^1$.

**Theorem 1.4** *Let $f$ be a $C^1$ diffeomorphism of a manifold $M$. If $(K, \gamma, h)$ is a $C^1$ Denjoy sub-system for $f$, then the unique $f$ invariant Borel probability measure that is supported on $\gamma(K)$ has at least one vanishing Lyapunov exponent.*

We easily deduce:

**Corollary 1.5** *Let $f$ be an area preserving $C^1$ diffeomorphism of a surface $M$. If $(K, \gamma, h)$ is a $C^1$ Denjoy sub-system for $f$, then the unique $f$ invariant Borel probability measure supported on $\gamma(K)$ has its two Lyapunov exponents equal to zero.*

*Remark 2*
- *(i) The second author proved in [6] that for a generic area preserving twist map, many Aubry-Mather sets are uniformly hyperbolic. Such an Aubry-Mather set cannot be a loop. Among them, those that have an irrational rotation number are Lipschitz Denjoy sub-systems but not $C^1$ Denjoy sub-systems.*
- *(ii) It was proved in [2] that any continuous invariant loop by a $C^{1+\alpha}$ diffeomorphism of a surface that contains no periodic point carries a unique invariant measure that has at least one zero Lyapunov exponent. The first remark shows that this result cannot be extended to Lipschitz Denjoy sub-systems that are not defined via an invariant curve.*
- *(iii) The first author proved in [1] a kind of reverse result of Corollary 1.5: if the Lyapunov exponents of a measure that is supported on an Aubry-Mather set with irrational rotation number of a symplectic twist map are zero, then the support of the measure is, in some sense, $C^1$ regular almost everywhere. The next proposition gives a counter-example in the non twist setting.*

**Proposition 1.6** *There exists a $C^1$ diffeomorphism $F$ of $\mathbb{T}\times\mathbb{R}$ that admits a Lipschitz Denjoy sub-system $(K, \gamma, h)$ such that*
- *the Lyapunov exponents of the unique invariant probability measure that is supported on $\gamma(\mathbb{T})$ are equal to zero;*
- *there exists no $C^1$ Denjoy sub-system $(K_0, \gamma_0, h_0)$ for a $C^1$ diffeomorphism $f$ of $\mathbb{T}\times\mathbb{R}$ such that $\gamma_0(K_0) = \gamma(K)$.*

*Remark 3 The diffeomorphism built in Proposition 1.6 has a Lipschitz foliation into invariant Lipschitz graphs on which the dynamics is a Denjoy counter-example.*

This notion of Denjoy sub-system can be extended to flows.

**Definition** 1.7 *Let $(\varphi_t)_{t\in\mathbb{R}}$ be a $C^k$ flow on a manifold $M$. A $C^k$* (resp. Lipschitz) Denjoy sub-flow for *$(\varphi_t)_{t\in\mathbb{R}}$ is a triplet $(K, j, (\psi_t)_{t\in\mathbb{R}})$ where*
- *$j : \mathbb{T}^2 \to M$ is a $C^k$ (resp. Lipschitz) embedding;*
- *$(\psi_t)_{t\in\mathbb{R}}$ is a $C^1$ flow of $\mathbb{T}^2$ with no periodic orbit nor dense orbit;*
- *if $K$ is the unique minimal invariant closed subset of $(\psi_t)_{t\in\mathbb{R}}$, then $j(K)$ is invariant by $(\varphi_t)_{t\in\mathbb{R}}$;*
- *for every $x \in K$ and every $t \in \mathbb{R}$, one has $j \circ \psi_t(x) = \varphi_t \circ j(x)$.*

**Theorem 1.8** *There exists no $C^2$ Denjoy sub-flow.*

In [4], a similar result for graphs is proved in the setting of weak K.A.M. theory.

## 2. Proof of Theorems 1.2 and 1.4

The key result, in the proofs of Theorems 1.2 et 1.4 is the following:

**Proposition 2.1** *Let $k \geq 1$ be some integer. Let $(K, \gamma, h)$ be a $C^k$ Denjoy sub-system for some $C^k$-diffeomorphism $f : M \to M$. There exists an orientation preserving $C^k$ diffeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ such that $\varphi_{|K} = h_{|K}$.*

*Proof.* With the notations of Proposition 2.1, observe that the two loops $\gamma : \mathbb{T} \to M$ and $f \circ \gamma : \mathbb{T} \to M$ are tangent along the $f$ invariant Cantor set $\gamma(K)$. Indeed, this set is included in both loops and has no isolated point.

Let $\mathcal{N}$ be a tubular neighborhood of $\gamma(\mathbb{T})$ in which we can define a $C^k$ projection $p : \mathcal{N} \to \gamma(\mathbb{T})$. Observe that $p \circ f \circ \gamma$, coinciding with $\gamma \circ h$ on $K$, is a $C^k$ embedding when restricted to a neighborhood of $K$. This implies that there exist two open neighborhoods $U$, $U'$ of $K$ in $\mathbb{T}$ such that $f \circ \gamma(U) \subset \mathcal{N}$ and such that $\gamma^{-1} \circ p \circ f \circ \gamma$ induces a $C^k$ diffeomorphism between $U$ and $U'$. The family of connected components of $U$ is an open covering of $K$. By compactness of $K$, there are finitely many componeets that meet $K$. So, taking a subset of $U$ if necessary, we can assume that the number of connected components of $U$ is finite, larger than 1, and that each connected component meets $K$. We can index these components, getting a family $(U_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ ordered in the usual cyclic order along $\mathbb{T}$. Setting $U_i' = \gamma^{-1} \circ p \circ f \circ \gamma(U_i)$, one gets another family $(U_i')_{i \in \mathbb{Z}/n\mathbb{Z}}$ ordered in the usual cyclic order along $\mathbb{T}$ because $\gamma^{-1} \circ p \circ f \circ \gamma_{|K} = h_{|K}$. We can find a covering $(V_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of $K$ by open and connected subsets of $\mathbb{T}$, satisfying $\overline{V_i} \subset U_i$ for every $i \in \mathbb{Z}/n\mathbb{Z}$. For every $j \in \mathbb{Z}/n\mathbb{Z}$, we define $V_i' = \gamma^{-1} \circ p \circ f \circ \gamma(V_i)$ and we denote $J_i$ the connected component of the complement of $\bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} V_i$ that lies between $V_i$ and $V_{i+1}$ and $J_i'$ the connected component of the complement of $\bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} V_i'$ that lies between $V_i'$ and $V_{i+1}'$. To extend $f_{|\bigcup_{1 \leq i \leq n} \overline{V}_i}$ to a $C^k$ diffeomorphism of $\mathbb{T}$, it is sufficient to find, for every $i \in \mathbb{Z}/n\mathbb{Z}$, an increasing $C^k$ diffeomorphism $\varphi_i : J_i \to J_i'$ with prescribed values of the derivatives $\varphi_i^{(l)}$, $1 \leq l \leq k$, at the two ends of $J_i$ to ensure a global differentiability. There is no obstruction to such a construction. □

Denjoy Theorem implies that under the assumption of Proposition 2.1, the integer $k$ must be equal to 1, so Theorem 1.2 is proved. Moreover, when $k = 1$, then $\mathbb{R}.\gamma'$ is a $Df$ invariant lines bundle along $\gamma(K)$ that satisfies $Df.\gamma' = \varphi' \, (\gamma' \circ \varphi)$. So, for every integer $n \geq 0$ one has

$$Df^n.\gamma' = (\varphi^n)' \, (\gamma' \circ \varphi^n).$$

We deduce that the unique $f$ invariant probability measure supported on $\gamma(K)$ has a Lyapunov exponent that coincides with the Lyapounov exponent of the unique invariant measure of $\varphi$, a $C^1$ diffeomorphism of $\mathbb{T}$. This Lyapunov exponent being known to vanish, Theorem 1.4 is proved.

## 3. Proof of Proposition 1.6

As explained in [5], one can construct a $C^1$ Denjoy counter-example $g : \mathbb{T} \to \mathbb{T}$ such that:

— $g$ has exactly two orbits of wandering intervals $(I_k)_{k\in\mathbb{Z}}$ and $(J_k)_{k\in\mathbb{Z}}$ where $\mathrm{Leb}(I_k) = \mathrm{Leb}(J_k) = \ell_k$;

— $\displaystyle\sum_{k\in\mathbb{Z}} \ell_k = \frac{1}{2}$;

— $\displaystyle\lim_{k\to\pm\infty} \frac{\ell_k}{\ell_{k+1}} = 1$;

— the derivative $g'$ is equal to 1 on the minimal compact invariant set $K$.

In particular, one has $\mathbb{T}\setminus K = \displaystyle\bigcup_{k\in\mathbb{Z}} \mathrm{Int}(I_k) \cup \bigcup_{k\in\mathbb{Z}} \mathrm{Int}(J_k)$.

There exists a Lipschitz function $\psi : \mathbb{T} \to \mathbb{R}$, uniquely defined up to an additive constant, such that

$$\psi'_{|\bigcup_{k\in\mathbb{Z}} \mathrm{Int}(I_k)} = 1,\ \ \psi'_{|\bigcup_{k\in\mathbb{Z}} \mathrm{Int}(J_k)} = -1.$$

Observe that $\psi'$ is constant along every orbit of $g$ that is contained in $\mathbb{T}\setminus K$.

The map $\eta : \mathbb{T} \to \mathbb{R}$ defined by $\eta = \psi\circ g - \psi$ is also Lipschitz and differentiable on $\mathbb{T}\setminus K$. Moreover, for every $\theta \in \mathbb{T}\setminus K$, one has

$$\eta'(\theta) = g'(\theta)\,\psi'(g(\theta)) - \psi'(\theta) = \psi'(\theta)\,(g'(\theta) - 1).$$

The fact that $\psi'$ is bounded on $\mathbb{T}\setminus K$ and that $1 - g'$ is continuous on $\mathbb{T}$ and vanishes on $K$ implies that $\eta$ is $C^1$.

Finally, let us define a $C^1$ diffeomorphism $F$ of $\mathbb{T}\times\mathbb{R}$ by the formula

$$F(\theta, r) = (g(\theta), r + \eta(\theta)).$$

The graph of $\psi$ is invariant by $F$ because

$$F(\theta, \psi(\theta)) = (g(\theta), \psi(\theta) + \eta(\theta)) = (g(\theta), \psi\circ g(\theta)),$$

Note that for every $\theta \in \mathbb{T}$ and every $k \in \mathbb{Z}$, one has

$$DF^k(\theta_0, \psi(\theta_0)) = \begin{pmatrix} (g^k)'(\theta_0) & 0 \\ b_k & 1 \end{pmatrix}$$

Hence the two Lyapunov exponents of the unique invariant measure that is supported on the graph of $\psi$ are zero.

Moreover, if $\theta \in K$, there exists two sequences $(I_{i_k})_{k\in\mathbb{N}}$ and $(J_{j_k})_{k\in\mathbb{N}}$ that converge to $\theta$. The slope of $\psi$ between the two ends of $I_{i_k}$ is 1 and the slope of $\psi$ between the two ends of $J_{j_k}$ is -1. This implies that there exists no $C^1$ Denjoy sub-system $(K_0, \gamma_0, h_0)$ of a diffeomorphism $f$ of $\mathbb{T}\times\mathbb{R}$ such that $\gamma_0(K_0) = \mathrm{graph}(\psi_{|K})$.

## 4. Proof of Theorem 1.8

We assume that $(K_0, j, (\psi_t)_{t\in\mathbb{R}})$ is a $C^2$ Denjoy sub-flow for some $C^2$ flow $(\varphi_t)_{t\in\mathbb{R}}$ on a manifold $M$. We choose on $\mathbb{T}^2$ an essential $C^2$ embedding $\gamma : \mathbb{T} \to \mathbb{T}^2$ that is transverse to the flow direction $\dot\psi$ of $(\psi_t)_{t\in\mathbb{R}}$. The first return map $r : \gamma(\mathbb{T}) \to \gamma(\mathbb{T})$ is then well defined, of class $C^1$, and the map $h = \gamma^{-1}\circ r\circ\gamma : \mathbb{T}\to\mathbb{T}$ is a $C^1$ Denjoy counter-example. We denote $K$ the minimal set of $h$ and set $\mathcal{K} = j\circ\gamma(K)$ and $\mathcal{T} = j\circ\gamma(\mathbb{T})$

Let $\mathcal{N}$ be a tubular neighborhood of $j(\mathbb{T}^2)$ in which we can define a $C^2$ projection $p : \mathcal{N} \to j(\mathbb{T}^2)$ and consider the $C^2$ hypersurface $\mathcal{H} = p^{-1}(\mathcal{T})$. The flow direction $\dot\varphi$ of $(\varphi_t)_{t\in\mathbb{R}}$ is transverse to $\mathcal{H}$ on $\mathcal{K}$. Moreover the first return map of $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ is well defined on $\mathcal{K}$ and coincide with $j\circ\gamma\circ h\circ\gamma^{-1}\circ j^{-1}{}_{|\mathcal{K}}$. One deduces that there exist two neighborhoods $\mathcal{W}$ and $\mathcal{W}'$ of $\mathcal{K}$ in $\mathcal{H}$ such that the first return map of

$(\varphi_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ is well defined on $\mathcal{W}$ and induces a $C^2$ diffeomorphism $\mathcal{R}$ between $\mathcal{W}$ and $\mathcal{W}'$. The map $\gamma^{-1}\circ j^{-1}{}_{|\mathcal{T}}\circ p\circ\mathcal{R}\circ j\circ\gamma$ is a $C^2$ map defined on a neighborhood of $K$ with values in $\mathbb{T}$ that coincides with the $C^1$ diffeomorphism $h$ on $K$. So it induces a $C^2$ diffeomorphism between two neighborhoods $U$, $U'$ of $K$. As in the proof of Proposition 2.1, we may require that $U$ and $U'$ have finitely many connected components and we can construct a $C^2$ diffeomorphims of $\mathbb{T}$ that coincides with $h$ in a neighborhood of $K$. This gives a contradiction.

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