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***Abstract*— In this paper, we consider the problem of minimizing the total time spent by a controlled dynamics outside a constraint set K. Also known as *time of crisis*, one essential feature of this problem is the discontinuity of the involved integral cost with respect to the state. We first relate this optimal control problem to a mixed initial-final problem with smooth data. Applying the classical theory of optimality conditions to the auxiliary (smooth) problem, we obtain as a main result second order necessary optimality conditions for the time of crisis. Considering the partition of the state space made out of K and its complementary, we notice that the problem can be seen as a particular case of a hybrid problem. Our analysis is thus a first step toward a second order analysis for the more general class of hybrid problems.**

I. Introduction

In various control problems, solutions to a controlled dynamics

$$\dot{x} = f(x, u), \tag{1}$$

starting at time 0 in a subset K of the state space must obey to state constraints

$$\forall t \geq 0,\ x(t) \in K, \tag{2}$$

where $K \subset \mathbb{R}^n$ ($n \geq 1$) is a non-empty closed subset and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Such a constraint set typically represents physical constraints to be satisfied along admissible trajectories of (1). It is well known that given $x_0 \in K$, there is a solution of (1) satisfying (2) if and only if

$$\forall x \in K,\ T_K(x) \cap F(x) \neq \emptyset, \tag{3}$$

where $T_K(x)$ denotes the contingent cone to K at point x (see [1], [2]) and $F(x) := \{f(x, u)\ ;\ u \in U\}$ is the velocity set. This result has been established in various contexts in the literature (see [7], [17] among others and [1] for exhaustive references on this subject).

It appears that in some application models, constraint (2) may not be satisfied at every time, in particular if (3) fails to hold. A simple example based on Lotka-Volterra's system highlights this phenomenon when the set K represents a threshold for a population not to be exceeded (see [3]). One may then wonder what is the "best" trajectory in terms of preserving as much as possible the state of the system in the set K. A possible approach to this question is to introduce the time of crisis as in [15] and to look for controls u for which the time spent by solutions of (1) outside K is minimal. This amounts to consider the following optimal control problem called *time of crisis*[1]:

$$\inf_{u(\cdot)\in\mathcal{U}} \int_0^{+\infty} \mathbb{1}_{K^c}(x_u(t, x_0))\ \mathrm{d}t, \tag{4}$$

where $\mathcal{U}$ is the set of admissible controls:

$$\mathcal{U} := \{u : [0, +\infty) \to U\ ;\ u \text{ meas.}\},$$

$x_u(\cdot, x_0)$ is the unique solution of (1) satisfying $x(0) = x_0$ at time 0, $\mathbb{1}_{K^c}$ is the characteristic function of the complementary in $\mathbb{R}^n$ of the set K, and $U \subset \mathbb{R}^m$ is non-empty. The consideration of (4) is also of great interest when the initial condition is not in the viability kernel of K under the dynamics f defined as (see [2]):

$$\mathrm{Viab}_K(f) := \{x_0 \in K\ ;\ \exists u \in \mathcal{U},\ \forall t \geq 0,\ x_u(t, x_0) \in K\}.$$

Problem (4) presents two essential features, namely, the infinite horizon and the discontinuity of $\mathbb{1}_{K^c}$ at the boundary of K. To our best knowledge, reducing (4) to a finite horizon when the infimum in (4) is finite is a complex question (in particular, chattering phenomenon at the boundary of K could occur for optimal trajectories). Nevertheless, whenever the viability kernel is non-empty, reachable from every state in $\mathbb{R}^n$, and if there is a uniform bound between two consecutive crossing times t_1 and t_2 (respectively from K to K^c and from K^c to K), then it can be shown that (4) is equivalent to the free terminal time problem

$$\inf_{u(\cdot)\in\mathcal{U}, T>0} \int_0^{T} \mathbb{1}_{K^c}(x_u(t, x_0))\ \mathrm{d}t\ \text{ s.t. } x_u(T) \in \mathrm{Viab}_K(f), \tag{5}$$

see [3]. Given this result, it seems relevant to consider, as a first approach, a variant of Problem (4) with a fixed and finite horizon $T > 0$, without terminal constraint. Over $[0, T]$, first order optimality conditions can be obtained via the hybrid maximum principle [11], [16] considering an extended hybrid dynamics obtained from (1) that takes into account if the state of the system is in K or in K^c. This principle has been used in [4], [5] to study a regularization scheme of the time of crisis.

The aim of the paper is to derive second order necessary optimality conditions for the time of crisis (formulated with a finite horizon) and for an optimal path which has a finite number of transverse crossing times. Our methodology is the following: we introduce a time transformation and an augmented dynamics, in the spirit of [12], [13], [14]. This enables us to define an auxiliary (smooth) optimal

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[1] In this context, when the state of the system is not in K, we say that the system is in "crisis".

control problem (P) of Mayer type with mixed initial-final conditions. Applying classical results to problem (P) and going back to the original one, we obtain that way optimality conditions for the time of crisis.

As already mentioned, the problem under study can be seen as a particular case of a hybrid problem (see [18]). To our best knowledge, second order optimality conditions have not been much addressed neither in the context of the time of crisis, nor in the context of hybrid problems (except [19]).

The paper is structured as follows. In section II, we give our main assumptions on the data for the second order analysis of the time crisis problem. In Section III, we state our main result (Theorem 3.1) about first and second order optimality conditions: it provides a first order optimality condition in Pontryagin's form as well as the positivity of the quadratic form associated to the time crisis in the critical cone in the case of a single crossing time for the optimal path. Next, we address the case of finite number of transverse crossing times (see Theorems 3.2-3.3). We summarize and explain in Section IV-C the main tools that are used to derive these optimality conditions in the case of a single crossing time (the methodology for obtaining the results in the case of multiple crossing times being the same, see Section IV-D). Because of brevity, we do not provide all the proofs of the results (which can be found in [6]).

II. Main assumptions

Throughout the rest of the paper $T > 0$ is fixed, $I := [0, T]$, n, m, and l are positive integers, and $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^s$ associated with the standard inner product written $a \cdot b$ for $a, b \in \mathbb{R}^s$ (s being a positive integer). We are given a closed subset K of $\mathbb{R}^n$ with non-empty interior, and we denote by $\mathrm{Int}(K)$, ∂K, and K^c the interior, the boundary, and the complementary of the set K. We suppose that the dynamics f fulfills the following (standard) assumptions:

- The mapping f is of class C^2 w.r.t. (x, u), and satisfies the linear growth condition: there exist $c_1 > 0$ and $c_2 > 0$ such that for all $x \in \mathbb{R}^n$ and all $u \in U$, one has:
$$|f(x,u)| \leq c_1|x| + c_2. \qquad (6)$$
- For any $x \in \mathbb{R}^n$, the velocity set $F(x) := \{f(x,u) \; ; \; u \in U\}$ is a non-empty compact convex subset of $\mathbb{R}^n$ where $U \subset \mathbb{R}^m$ is closed and non-empty.

Under these assumptions, for any $x_0 \in \mathbb{R}^n$, there is a unique solution $x_u(\cdot)$ of the Cauchy problem

$$\left\{ \begin{array}{ll} \dot{x} & = f(x,u), \\ x(0) & = x_0, \end{array} \right. \qquad (7)$$

defined over $[0, T]$. In the sequel, we focus on the following optimal control problem:

$$\inf_{u \in \mathcal{U}} J_T(u) := \int_0^T \mathbb{1}_{K^c}(x_u(t)) \; \mathrm{d}t. \qquad (8)$$

By an optimal solution of (8), we mean a (global) optimal control $u \in \mathcal{U}$ of (8). Existence of an optimal solution for (8) is standard (we refer to [5], [15]).

To express optimality conditions, it is convenient to write U and K as sub-level sets of given functions satisfying qualification conditions. We fix for the rest of the article a solution $\bar{u} \in \mathcal{U}$ to (8), with associated trajectory $\bar{x} := x_{\bar{u}}$, satisfying Assumption (H1).

(H1) There is a function $c : \mathbb{R}^m \to \mathbb{R}^l$ of class C^2 such that
$$U = \{u \in \mathbb{R}^m \; ; \; c_i(u) \leq 0, \; 1 \leq i \leq l\}. \qquad (9)$$
For $\delta > 0$ and $i \in \{1, ..., l\}$, we define $\Delta^\delta_{c,i} := \{t \in (0, T) \, ; \, c_i(\bar{u}(t)) \leq -\delta\}$ and for $t \in (0, T)$, let
$$I^\delta_c(t) := \{i \in \{1, ..., l\} \, ; \, t \in \Delta^\delta_{c,i}\}.$$
Given a subset $J = \{i_1, ..., i_{|J|}\} \subseteq \{1, ..., l\}$ of cardinality $|J|$, we set $c_J(u) := (c_{i_1}(u), ..., c_{i_{|J|}}(u)) \in \mathbb{R}^{|J|}$. We assume that there exist $\varepsilon > 0$ and $\delta > 0$ such that for a.e. $t \in [0, T]$:
$$\varepsilon|\xi| \leq \nabla c_{I^\delta_c(t)}(\bar{u}(t))\xi, \quad \forall \xi \in \mathbb{R}^{|I^\delta_c(t)|}. \qquad (10)$$

Remark 2.1: Inequality (10), referred to as *linear independence of gradients of active constraints* condition is classical. It implies the following properties (see, *e.g.*, [9]):

- Inward pointing condition: there exist $\varepsilon > 0$ and $v \in L^\infty(0, T; \mathbb{R}^m)$ such that
$$c(\bar{u}(t)) + Dc(\bar{u}(t))v(t) \leq -\varepsilon \quad \text{a.e. } t \in (0, T). \quad (11)$$
- There exists $\delta > 0$ such that the following mapping from $\in L^2(0, T; \mathbb{R}^m)$ into $L^2(\Delta^\delta_{c,i})$ is onto:
$$v \mapsto \Big((Dc_i(\bar{u}(\cdot))v(\cdot))_{|\Delta^\delta_{c,i}} \Big)_{i=1,\ldots,l}. \qquad (12)$$

Note that the inward pointing condition ensures the existence of a Lagrange multiplier in $L^\infty(0, T; \mathbb{R}^l)$ for (8) under the control constraint $c(u) \leq 0$ (see also [8], [9], [10]).

Throughout the article, we also assume that K satisfies the following hypothesis.

(H2) There is a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that
$$K = \{x \in \mathbb{R}^n \; ; \; g(x) \leq 0\}. \qquad (13)$$

The analysis of optimal controls of (8) and associated trajectories relies on the notion of *crossing time*.

Definition 2.1: (i) A crossing time from K to K^c is a time $t_c \in (0, T)$ for which there is $\varepsilon > 0$ such that for any time $t \in (t_c - \varepsilon, t_c]$ (resp. $t \in (t_c, t_c + \varepsilon)$) one has $\bar{x}(t) \in K$ (resp. $\bar{x}(t) \in K^c$).
(ii) A crossing time t_c from K to K^c is *transverse* if the control $\bar{u}$ is right- and left- continuous at time t_c, and if
$$\dot{\bar{x}}(t_c^\pm) \cdot \nabla g(\bar{x}(t_c)) \neq 0. \qquad (14)$$

Note that there are similar definitions for crossing times from K^c to K and transverse crossing times from K^c to K. The analysis that we carry out in this paper relies on the following assumption on $\bar{x}$:

(H3) The optimal trajectory $\bar{x}$ possesses exactly $r \in \mathbb{N}^*$ transverse crossing times $\bar{\tau}_1 < \cdots < \bar{\tau}_r$ in $(0, T)$ such that $\bar{\tau}_{2i+1}$ (resp. $\bar{\tau}_{2i}$) is a crossing time from K to K^c (resp. from K^c to K). For all $t \in [0,T]\backslash\{\bar{\tau}_1, \cdots, \bar{\tau}_r\}$, $g(\bar{x}(t)) \neq 0$.

Remark 2.2: Assumption (H3) implicitly supposes that the initial condition satisfies $x_0 \in \mathrm{Int}(K)$, but we could consider as well x_0 in K^c with slight modifications. It also excludes the chattering phenomenon at the boundary of K.

III. Optimality conditions

In this section, we provide first and second order optimality conditions for (8).

A. The case of a single crossing time

We recall that $\bar{u}$ is a fixed solution to Problem (8) with associated trajectory $\bar{x} = x_{\bar{u}}$, satisfying (H1). We assume in this subsection that (H3) is satisfied with $r = 1$; the unique crossing time is denoted by $\bar{\tau}$.

Let $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be the Hamiltonian:

$$H(x,p,u) := p \cdot f(x,u),$$

and $H_a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ the augmented Hamiltonian defined as

$$H^a(x,p,u,\nu) := p \cdot f(x,u) + \nu \cdot c(u). \tag{15}$$

We start by introducing Lagrange and Pontryagin multipliers.

Definition 3.1: A triplet $(\alpha, \gamma, \nu) \in \mathbb{R}_+ \times \mathbb{R} \times L^\infty(0,T;\mathbb{R}^l)$ is called *Lagrange multiplier* (associated with $\bar{u}$ and problem (8)) if the following conditions are satisfied:

- The triplet (α, γ, ν) is non-zero and the Lagrange multiplier ν satisfies the following sign and complementarity conditions:

$$\nu(t) \geq 0, \quad \nu(t) \cdot c(\bar{u}(t)) = 0, \quad \text{a.e. } t \in [0,T]. \tag{16}$$

- There exists a function p: $[0,T] \to \mathbb{R}^n$, whose restrictions to $[0,\bar{\tau})$ and $(\bar{\tau},T]$ are absolutely continuous, which satisfies the following adjoint equation

$$\begin{aligned} \dot{p}(t) &= -\nabla_x H(\bar{x}(t), p(t), \bar{u}(t)) \quad \text{a.e. } t \in [0,T], \\ p(T) &= 0, \end{aligned} \tag{17}$$

and the following jump condition at $\bar{\tau}$:

$$p(\bar{\tau}^+) - p(\bar{\tau}^-) = \gamma \nabla g(\bar{x}(\bar{\tau})). \tag{18}$$

- The augmented Hamiltonian is stationary w.r.t. u:

$$\nabla_u H^a(\bar{x}(t), p(t), \bar{u}(t), \nu(t)) = 0 \text{ a.e. } t \in [0,T]. \tag{19}$$

- The following relation holds true

$$\int_0^T \rho_{\bar{\tau}}(t) H(\bar{x}(t), p(t), \bar{u}(t)) \ \mathrm{d}t = \alpha. \tag{20}$$

where ρ_τ is the function defined as:

$$\rho_\tau(t) := \frac{1}{\tau} \quad \text{if } t \in (0,\tau), \quad \rho_\tau(t) := \frac{-1}{T-\tau} \quad \text{if } t \in (\tau, T).$$

Definition 3.2: We call Pontryagin multiplier a Lagrange multiplier (α, γ, ν) satisfying *Pontryagin's Principle*, *i.e*, for the associated costate p, one has a.e. in I

$$H(\bar{x}(t), p(t), \bar{u}(t)) \leq H(\bar{x}(t), p(t), u), \quad \forall u \in U. \tag{21}$$

We denote by $\Lambda_L(\bar{u}, \bar{\tau})$ and $\Lambda_P(\bar{u}, \bar{\tau})$ the sets of Lagrange and Pontryagin multipliers associated with $\bar{u}$ and Problem (8).

Lemma 3.1: The set of Pontryagin multipliers $\Lambda_P(\bar{u}, \bar{\tau})$ is non-empty.

See Section IV-C for the proof. To state second order optimality conditions, let us introduce the linearized dynamics, the critical cone $C(\bar{u}, \bar{\tau})$, and the quadratic form $\Omega(\bar{u}, \bar{\tau})$ associated with (8). Given $(\delta u, \delta\tau) \in L^2(0,T;\mathbb{R}^m) \times \mathbb{R}$, we consider the following *linearized system*:

$$\begin{aligned} &\tfrac{d}{dt}\delta x(t) = Df[t](\delta x(t), \delta u(t)) + \rho_{\bar{\tau}}(t)\delta\tau f[t], \text{ a.e. } t \in I, \\ &\delta x(0) = 0. \end{aligned} \tag{22}$$

We use the notation $[t]$ as a shortening of $(\bar{x}(t), \bar{u}(t))$.

Definition 3.3: For a solution δx to (22), let us introduce the conditions

$$Dg(\bar{x}(\tau))\delta x(\bar{\tau}) = 0, \tag{23}$$

$$c_i(\bar{u}(t)) = 0 \Rightarrow Dc_i(\bar{u}(t))\delta u(t) = 0, \ i = 1,...,l, \tag{24}$$

where (24) holds a.e. over $(0,T)$. The *critical cone* $C(\bar{u}, \bar{\tau})$ is then defined as the set of pairs $(\delta u, \delta\tau) \in L^2(0,T;\mathbb{R}^{2l}) \times \mathbb{R}$ such that the solution to (22) satisfies (23)-(24).

Given $(\alpha, \gamma, \nu) \in \Lambda_L(\bar{u}, \bar{\tau})$ and $(\delta u, \delta\tau) \in L^2(0,T;\mathbb{R}^m) \times \mathbb{R}$, we define *the quadratic form* $\Omega[\alpha,\gamma,\nu](\delta u, \delta\tau)$ as follows:

$$\begin{aligned} \Omega[\alpha,\gamma,\nu](\delta u, \delta\tau) := {} & \gamma D^2 g(\bar{x}(\bar{\tau}))(\delta x(\bar{\tau}))^2 \\ & + 2\delta\tau \int_0^T \rho_{\bar{\tau}}(t) DH[t](\delta x(t), \delta u(t)) \ \mathrm{d}t \\ & + \int_0^T D^2 H^a[t](\delta x(t), \delta u(t))^2 \ \mathrm{d}t, \end{aligned}$$

where δx denotes the solution to (22), p is the unique solution of (17) (uniquely defined from γ thanks to (17)-(18)), and $[t]$ is a shortening of $(\bar{x}(t), p(t), \bar{u}(t), \nu(t))$. Like before, the first- and second-order derivatives of the Hamiltonians must be considered with respect to (x,u) only. The second-order necessary optimality conditions in Pontryagin form for the time crisis problem with one crossing time are given by the following theorem.

Theorem 3.1: For all $(\delta u, \delta\tau) \in C(\bar{u}, \bar{\tau})$, there exists $(\alpha, \gamma, \nu) \in \Lambda_P(\bar{u}, \bar{\tau})$ such that

$$\Omega[\alpha,\gamma,\nu](\delta u, \delta\tau) \geq 0.$$

We finally have the following result, dealing with the non-singularity of Pontryagin multipliers.

Lemma 3.2: For all $(\alpha, \gamma, \nu) \in \Lambda_P(\bar{u}, \bar{\tau})$, $\alpha > 0$. Moreover, there is a unique Pontryagin multiplier such that $\alpha = 1$.

We refer to [6] for a proof.

B. The case of multiple crossing times

We suppose in this subsection that (H3) is satisfied with crossing points $\bar{\tau}_1 < ... < \bar{\tau}_r$. We denote by $\bar{\tau} \in (0,T)^r$ the vector $(\bar{\tau}_1, ..., \bar{\tau}_r)$ and make use of the conventions $\bar{\tau}_0 = 0$ and $\bar{\tau}_{r+1} = T$.

For the generalization of the first- and second-order optimality conditions, we re-define the mapping ρ_τ as a mapping in $L^\infty(0,T;\mathbb{R}^r)$ as follows:

$$\begin{array}{lll} (\rho_\tau(t))_j & = \frac{1}{\tau_j - \tau_{j-1}} & \text{if } t \in (j-1, j), \\ (\rho_\tau(t))_j & = \frac{-1}{\tau_{j+1} - \tau_j} & \text{if } t \in (j, j+1), \\ (\rho_\tau(t))_j & = 0 & \text{otherwise.} \end{array}$$

We begin with first order optimality conditions and let $H_0 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be defined by

$$H_0(x,p,u) := p \cdot f(x,u) + \mathbb{1}_{K^c}(x).$$

We also note that Lemma 3.2 is still valid in the situation with several crossing times, thus the optimality conditions can be formulated for the unique Pontryagin multiplier satisfying $\alpha = 1$.

Theorem 3.2: There exists a unique pair $(\gamma, \nu) \in \mathbb{R}^r \times L^\infty(0,T;\mathbb{R}^l)$ satisfying the following properties:

- The Lagrange multiplier ν satisfies (16).
- There exists a function $p\colon [0,T] \to \mathbb{R}^n$, whose restrictions to $[0,\bar{\tau}_1)$, $(\bar{\tau}_1, \bar{\tau}_2)$,...,$(\bar{\tau}_r, T]$ are absolutely continuous, which satisfies the following adjoint equation

$$\begin{array}{ll} \dot{p}(t) & = -\nabla_x H(\bar{x}(t), p(t), \bar{u}(t)) \quad \text{a.e. } t \in [0,T], \\ p(T) & = 0, \end{array} \tag{25}$$

  and the jump conditions at the crossing times $\bar{\tau}_j$:

$$p(\bar{\tau}_j^+) - p(\bar{\tau}_j^-) = \gamma_j \nabla g(\bar{x}(\bar{\tau}_j)),\ 1 \le j \le r. \tag{26}$$

- The augmented Hamiltonian H_a is stationary w.r.t. v, i.e., it satisfies (19).
- The following relation holds true for $j = 1, ..., r$:

$$\int_0^T (\rho_{\bar{\tau}}(t))_j H(\bar{x}(t), p(t), \bar{u}(t))\ \mathrm{d}t + (-1)^j = 0. \tag{27}$$

Moreover, the mapping $t \in (0,T) \mapsto H_0(\bar{x}(t), p(t), \bar{u}(t))$ is constant almost everywhere.

For writing second order optimality conditions, we need to re-define the linearized dynamics, the critical cone associated with the constraints, as well as the quadratic form Ω.

The linearized dynamics, for $\delta u \in L^\infty(0,T;\mathbb{R}^m)$ and $\delta\tau \in \mathbb{R}^r$ reads:

$$\begin{array}{l} \dfrac{d}{dt}\delta x(t) = Df[t](\delta x(t), \delta u(t)) + (\rho_{\bar{\tau}}(t) \cdot \delta\tau) f[t] \text{ a.e. } t \in I, \\ \delta x(0) = 0. \end{array} \tag{28}$$

Definition 3.4: For a solution δx to (28), let us introduce the conditions

$$\sum_{j=1}^r (-1)^j \delta\tau_j \le 0, \tag{29}$$

$$Dg(\bar{x}(\bar{\tau}_j))\delta x(\bar{\tau}_j) = 0,\ j = 1, ..., r, \tag{30}$$

$$c_i(\bar{u}(t)) = 0 \Longrightarrow Dc_i(\bar{u}(t))\delta u(t) = 0,\ i = 1, ..., l \tag{31}$$

where (31) holds a.e. over $(0,T)$. The critical cone $C(\bar{u}, \bar{\tau})$ is then defined as the set of pairs $(\delta u, \delta\tau) \in L^2(0,2;\mathbb{R}^{2l}) \times \mathbb{R}^r$ such that the solution to (28) satisfies (29)-(30)-(31).

The quadratic form $\Omega(\delta u, \delta\tau)$ is defined as

$$\begin{aligned} \Omega(\delta u, \delta\tau) :=& \sum_{j=1}^r \gamma_j D^2 g(\bar{x}(\tau_j))(\delta x(\bar{\tau}_j))^2 \\ &+ \int_0^T D^2 H^a[t](\delta x(t), \delta u(t))^2\ \mathrm{d}t \\ &+ 2\int_0^T (\rho_{\bar{\tau}}(t) \cdot \delta\tau) DH[t](\delta x(t), \delta u(t))\ \mathrm{d}t. \end{aligned} \tag{32}$$

Second order optimality for the time of crisis is given by the next theorem.

Theorem 3.3: For every $(\delta u, \delta\tau) \in C(\bar{u}, \bar{\tau})$, one has $\Omega(\delta u, \delta\tau) \ge 0$.

IV. A RELATED MAYER CONTROL PROBLEM

In this section, we provide the main elements of proof of Lemma 3.1 and Theorem 3.1. The main idea is to construct a weak solution to a smooth optimal control problem (P), for which available results can be applied. Problem (P) is obtained with a time transformation and an augmentation of the dynamics, described in the following two subsections in the case of a single crossing time. We deal with the case of multiple crossing times in subsection IV-D.

A. Time transformation

We assume that (H3) holds with $r = 1$; the unique crossing time is denoted $\bar{\tau}$. Let us first introduce a time transformation as follows. For $\tau \in (0,T)$, let $\pi_\tau : [0,2] \to [0,T]$, $s \mapsto t := \pi_\tau(s)$ be the piecewise-affine function defined as

$$\pi_\tau(s) := \begin{cases} \tau s, & \text{if} \quad s \in [0,1], \\ (T-\tau)s + 2\tau - T, & \text{if} \quad s \in [1,2]. \end{cases} \tag{33}$$

It is easily seen that the change of variable π_τ is one-to-one if and only if $\tau \in (0,T)$. Given $u \in \mathcal{U}$, we set

$$\left| \begin{array}{ll} \tilde{u}(s) & := u(\pi_\tau(s)), \\ \tilde{x}(s) & := x(\pi_\tau(s)), \end{array} \right. \quad s \in [0,2], \tag{34}$$

where x denotes the unique solution of (7) associated with u. The trajectory $\tilde{x}$ is the unique solution to the ODE

$$\begin{cases} \dfrac{d\tilde{x}}{ds}(s) & = \dfrac{d\pi_\tau}{ds}(s) f(\tilde{x}(s), \tilde{u}(s)) \quad \text{a.e. } s \in [0,2]. \\ \tilde{x}(0) & = x_0, \end{cases} \tag{35}$$

We can consider now the following set of admissible controls

$$\tilde{\mathcal{U}} := \{\tilde{u} : [0,2] \to U \ ;\ \tilde{u} \text{ meas.}\},$$

and the following optimal control problem:

$$\inf_{\tilde{u} \in \tilde{\mathcal{U}},\ \tau \in (0,T)} T - \tau \quad \text{s.t. } g(\tilde{x}_{\tilde{u},\tau}(1)) = 0, \tag{36}$$

where $\tilde{x}_{\tilde{u},\tau}$ is the unique solution of (35). Let us emphasize the fact that τ is an optimization variable of the problem, involved in the dynamics of the system. The crossing time of the trajectory is fixed to 1. We adopt the following definition of minimum.

Definition 4.1: A pair $(\tilde{u}, \tau) \in \tilde{\mathcal{U}} \times (0, T)$ is a weak minimum of (36) if there exists $\varepsilon > 0$ such that for all control $\tilde{u}' \in \tilde{\mathcal{U}}$ and all $\tau' \in (0, T)$ one has:

$$\|\tilde{u}' - \tilde{u}\|_{L^\infty(0,2;\mathbb{R}^m)} \leq \varepsilon \quad \text{and} \quad |\tau - \tau'| \leq \varepsilon \quad \Rightarrow \quad T - \tau \leq T - \tau'. \tag{37}$$

The next proposition is a key result to reformulate (8) as a classical optimal control problem.

Proposition 4.1: (i) Let $\tilde{u} := \bar{u} \circ \pi_{\bar{\tau}}$. Then, $(\tilde{u}, \bar{\tau})$ is a weak minimum of (36).
(ii) For all $\omega \in (0, 1)$, there exists $\varepsilon > 0$ such that for all $(\tilde{u}', \tau') \in \tilde{\mathcal{U}} \times (0, T)$, one has:

$$\begin{cases} \|\tilde{u} - \tilde{u}'\|_{L^\infty(1-\omega,1+\omega;\mathbb{R}^m)} \leq \varepsilon \\ \|\tilde{u} - \tilde{u}'\|_{L^1(0,1-\omega;\mathbb{R}^m)} \leq \varepsilon \\ \|\tilde{u} - \tilde{u}'\|_{L^1(1+\omega,2;\mathbb{R}^m)} \leq \varepsilon \\ |\bar{\tau} - \tau'| \leq \varepsilon \end{cases} \Rightarrow T - \tau \leq T - \tau',$$

Proof: The key idea in the proof of this proposition is to show that for a control $\tilde{u}'$ satisfying the above inequalities, then the associated trajectory $\tilde{x}_{\tilde{u}',\tau}$ also possesses a unique crossing time, at time $s = 1$. This amounts to show that the mapping

$$t \mapsto g(\tilde{x}_{\tilde{u}',\tau}(t))$$

is negative over $[0, 1)$ and positive over $(1, 2]$. Because the nominal trajectory has a unique transverse crossing time, and since its associated control is left and right continuous at the crossing time, the result follows using continuity arguments and the continuity of the input-output mapping (see more details in [6]). ■

B. Augmentation of the dynamics and the new Problem (P)

The goal now is to formulate (36) over the fixed interval $[0, 1]$ to avoid the use of the intermediate condition $g(\tilde{x}_{\tilde{u},\tau}(1)) = 0$ (which will be replaced by an initial-final time condition), and so that we can use classical results of optimal control theory. Hereafter, we use the notation

$$y := \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \xi \end{bmatrix}, \quad v := \begin{bmatrix} v^{(1)} \\ v^{(2)} \end{bmatrix}.$$

for vectors in $\mathbb{R}^{2n+1}$ and in $\mathbb{R}^{2m}$ respectively. Consider the mappings $F : \mathbb{R}^{2n+1} \times \mathbb{R}^{2m} \to \mathbb{R}^{2n+1}$ (standing for an augmented dynamics) and $G : \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+2}$ (standing for a mixed initial-final constraint) defined respectively as

$$F(y, v) := \begin{bmatrix} \xi f(y^{(1)}, v^{(1)}) \\ (T - \xi) f(y^{(2)}, v^{(2)}) \\ 0 \end{bmatrix},$$

and

$$G(y_0, y_1) := \begin{bmatrix} y_0^{(1)} \\ \xi_0 \\ y_0^{(2)} - y_1^{(1)} \\ g(y_1^{(1)}) \end{bmatrix},$$

where $y_0 := (y_0^{(1)}, y_0^{(2)}, \xi_0)$ and $y_1 := (y_1^{(1)}, y_1^{(2)}, \xi_1)$. In this setting, the set of admissible controls is

$$\mathcal{V} := \left\{ v := (v^{(1)}, v^{(2)}) : [0, 1] \to U \times U \; ; \; v \text{ meas.} \right\},$$

and we also define the set

$$C := \{x_0\} \times (0, T) \times \{0_{\mathbb{R}^n}\} \times \{0\} \subset \mathbb{R}^{2n+2}.$$

Remark 4.1: The set C comprises the initial condition at time 0, the fact that $\tau \in (0, T)$ is free, the continuity of the trajectory at time τ, and finally, the fact that the trajectory lies on the boundary of K at time τ

The controlled dynamics then becomes

$$\frac{dy}{ds}(s) = F(y(s), v(s)), \tag{38}$$

with $v \in \mathcal{V}$ and $s \in [0, 1]$. We denote by $\mathcal{T}$ the set of pairs (y, v) satisfying (38), with $v \in \mathcal{V}$. Finally, we define a terminal pay-off $\psi : \mathbb{R}^{2n+1} \to \mathbb{R}$ of class C^2 as

$$\psi(y) = T - \xi.$$

The new optimal control problem reads as follows:

$$\inf_{(y,v)\in\mathcal{T}} \psi(y(1)) \quad \text{s.t. } G(y(0), y(1)) \in C. \tag{P}$$

Note that we keep the variable y as an optimization variable, since its initial condition is not prescribed anymore and thus y cannot be expressed as a function of the control v. Problem (P) is smooth : it falls into the class of Mayer problems with mixed initial-terminal constraints.

Let us now recall the definition of a weak and Pontryagin minima for (P).

Definition 4.2: A pair $(\bar{y}, \bar{v}) \in \mathcal{T}$ is a weak minimum (resp. a Pontryagin minimum) of (P) if $G(\bar{y}(0), \bar{y}(1)) \in C$ and if there exists $\varepsilon > 0$ such that for all $(y, v) \in \mathcal{T}$ satisfying $G(y(0), y(1)) \in C$, one has:

$$|y(0) - \bar{y}(0)| \leq \varepsilon \quad \text{and} \quad \|v - \bar{v}\|_{L^r(0,1;\mathbb{R}^{2m})} \leq \varepsilon \;\; \Rightarrow \;\; \psi(\bar{y}(1)) \leq \psi(y(1)), \tag{39}$$

for $r = \infty$ (resp. $r = 1$).

Since Problem (P) has a classical structure, we only need now to relate a local solution to (P) to a solution of the time crisis. This is done in the following proposition.

Proposition 4.2: The pair $(\bar{y}, \bar{v}) \in \mathcal{V}$, defined as follows, is a weak minimum of (P):

$$\bar{y}(s) := \left| \begin{array}{l} \bar{y}^{(1)}(s) := \tilde{x}(s), \\ \bar{y}^{(2)}(s) := \tilde{x}(s + 1), \\ \bar{\xi}(s) := \bar{\tau}, \end{array} \right. \tag{40}$$

$$\bar{v}(s) := \left| \begin{array}{l} \bar{v}^{(1)}(s) := \tilde{u}(s), \\ \bar{v}^{(2)}(s) := \tilde{u}(s + 1), \end{array} \right. \tag{41}$$

where $s \in [0, 1]$, $\bar{\tau}$ is the unique crossing time of $\bar{x}$, $\tilde{u} = \bar{u} \circ \pi_{\bar{\tau}}$, and $\tilde{x} = \bar{x} \circ \pi_{\bar{\tau}}$.

Proof: This follows from Proposition 4.1 and the above transformations on the dynamics. ■

C. Proof of the optimality conditions (Theorem 3.1)

By Proposition 4.2, the pair $(\bar{y}, \bar{v}) \in \mathcal{V}$ is a weak minimum of (P), which is an optimal control problem with smooth data. Therefore, first and second order necessary optimality conditions can be applied. We refer the reader to [8, Theorem 4.9]. The technical assumptions needed for the application of this result follow here from (11) and (12). By performing a backward transformation to the one introduced in the first two subsections, we obtain the following result: the set $\Lambda_L(\bar{u}, \bar{\tau})$ is not empty, moreover, for all $(\delta u, \delta\tau) \in C(\bar{u}, \bar{\tau})$ there exists $(\alpha, \gamma, \nu) \in \Lambda_L(\bar{u}, \bar{\tau})$ such that $\Omega[\alpha, \gamma, \mu](\delta u, \delta\tau) \geq 0$.

Let us mention that the jump condition (18) is deduced from the transversality condition satisfied by $(\bar{y}, \bar{v})$.

The statement of Theorem 3.1 is however stronger, since it involves Pontryagin multipliers. For obtaining this result (that is, Lemma 3.1), one needs to prove that $(\bar{y}, \bar{u})$ satisfies optimality conditions in Pontryagin form. This is not obvious, since $(\bar{y}, \bar{u})$ is not known to be a Pontryagin minimum for Problem (P). This difficulty can be overcome by introducing a mixed notion of minima (between weak and Pontryagin minima), corresponding to the perturbations allowed in Proposition 4.1(ii) for a given value of $\omega \in (0, 1)$. This mixed notion yields first and second order optimality conditions with multipliers satisfying Pontryagin's principle a.e. in $(0, 1)$, except on an interval of size 2ω. Roughly speaking, our optimality conditions in Pontryagin form can finally be obtained by passing to the limit when ω tends to 0; once again, we refer to [6] for the technical details. This yields Lemma 3.1 and concludes the proof of Theorem 3.1.

D. Time transformation in the case of multiple crossing times

The change of variable and dynamics follow the same scheme as in the case of one crossing time. One essentially needs to apply the previous techniques at each crossing time and to introduce a problem in dimension $(r + 1)n + 1$ in place of $2n + 1$. We shall next only give the counter part of (36) in this setting (for more details, see [6]).

We first need to define a new change of variables adapted to the r crossing times. Given $\tau \in (0, T)^r$, we define the mapping $\pi_\tau \colon s \in [0, r + 1] \to [0, T]$ as follows:

$$\pi_\tau(s) := \tau_j + (s - j)(\tau_{j+1} - \tau_j),\ j = 0, ..., r,\ s \in [j, j + 1].$$

Given a control $\tilde{u} \in L^\infty(0, r + 1; U)$ and $\tau \in (0, T)^r$, there is a unique solution $\tilde{x}_{\tilde{u},\tau}$ of the Cauchy problem

$$\begin{cases} \dfrac{d\tilde{x}}{ds}(s) &= \dfrac{d\pi_\tau}{ds}(s) f(\tilde{x}(s), \tilde{u}(s)) \quad \text{a.e. } s \in [0, r + 1], \\ \tilde{x}(0) &= x_0. \end{cases}$$

The optimal control problem to be considered, after change of variable is now

$$\inf_{\tilde{u} \in \tilde{\mathcal{U}},\ \tau \in (0,T)^r} \sum_{j=1}^{r} (-1)^j \tau_j \quad \text{s.t. } g(\tilde{x}_{\tilde{u},\tau}(j)) = 0,\ j = 1, ..., r.$$

Again, we are in a position to apply first-and second-order optimality conditions for the extension of this problem in dimension $(r + 1)n + 1$ as for Problem (P). Theorems 3.2-3.3 are obtained going back to the original variables.

V. Conclusion

We have presented techniques (such as a time transformation and an augmentation of dynamics allowing to use optimality conditions in a smooth framework) that provide first and second order optimality conditions for the time of crisis problem. As already mentioned, it is closely related to a hybrid optimal control problem. Hence, future works could investigate how to apply those techniques in the more general hybrid framework. Our analysis of optimality conditions for the time of crisis also relies on a transverse hypothesis on optimal paths. It could be of interest to find classes of systems for which solutions possess this structure, or to provide optimality conditions without this assumption.

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