

# Tangency property and prior-saturation points in planar minimal time problems

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**Abstract:** In this paper, we address properties of the minimal time synthesis for control-affine-systems in the plane involving a saturation point for the singular control. First, we provide sufficient conditions on the data ensuring occurrence of a prior-saturation point. Then, we show that the bridge (*i.e.*, the optimal bang arc issued from the singular locus at this point) is tangent to the switching curve at the prior-saturation point. We illustrate these results on a fed-batch model in bioprocesses.

*Keywords:* Geometric optimal control, Minimum time problems, Singular arcs.

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## 1. INTRODUCTION

In this paper, we consider minimal time problems governed by single-input control-affine-systems in the plane

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)), \quad |u(t)| \leq 1,$$

where  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are smooth vector fields. Syntheses for such problems have been investigated a lot in the literature (see, *e.g.*, Bonnard and Chyba (2003); Boscaïn and Piccoli (2004); Piccoli (1996); Sussmann (1987)). An exhaustive description of the various encountered singularities can be found in Boscaïn and Piccoli (2004), as well as an algorithm leading to the determination of optimal paths. It is worth mentioning that even though many techniques exist in this setting, the computation of an optimal feedback synthesis (global) remains in general difficult because of the occurrence of geometric loci such as singular arcs, switching curves, cut-loci...

Our aim in this work is to study properties of the minimal time synthesis when saturation occurs on a singular arc<sup>1</sup>. Saturation means that the singular control  $u_s$  (which allows the associated trajectory to stay on the singular locus) becomes non admissible, that is,  $|u_s| > 1$ . Such a situation naturally appears in several application models, see, *e.g.*, Bayen et al. (2017, 2015); Ledzewicz and Sch attler (2007); Rapaport et al. (2016); Bakir et al. (2019). The occurrence of such a phenomenon implies the following (non-intuitive) property that, if a singular arc is optimal, then it should leave the singular locus at a so-called *prior-saturation point* before reaching the saturation point. This property has been studied in the literature in various situations such as for control-affine systems in dimension 2 and 4 (see, *e.g.*, Sch attler and Jankovic (1993); Sch attler and Ledzewicz (2012); Bonnard and De Morant (1995); Bayen et al. (2015) and references herein).

Our main goal in this paper is to provide new qualitative properties on the minimum time synthesis when saturation occurs:

- We shall first give a set of conditions that guarantee occurrence of a prior-saturation point showing that the system leaves the singular arc at this point (before reaching the saturation point) with the maximal value for the control, see Proposition 5. This last arc is usually called *bridge* following the terminology as in Bonnard et al. (2012, 2015) (see also Bonnard and Pelletier (1995); Bonnard and Chyba (2003)).
- We shall also introduce a shooting function that allows an effective computation of the prior-saturation point. This mapping is used to show our main result (Theorem 15) which can be stated as follows: when the system exhibits a switching curve emanating from the prior-saturation point, then this curve is tangent to the bridge (in the cotangent bundle) at this point.

The tangency property (in the state space) has been pointed out in several application models (see, *e.g.*, Bayen et al. (2015); Bonnard et al. (2012)). To the best of our knowledge, this fundamental property in the synthesis has not been addressed previously in this general setting in the literature.

The paper is structured as follows: in Section 2, we introduce the saturation phenomenon via Pontryagin's Principle. In Section 3, we provide a set of conditions involving the target set and the system ensuring occurrence of the prior-saturation phenomenon. In Section 4, we show the tangency property between the switching curve emanating from a prior-saturation point and the bridge. Finally, we depict this geometrical property in Section 5 for a fed-batch model Moreno (1999); Bayen et al. (2015). Because of brevity, some proofs of the results presented hereafter can be found in Bayen and Cots (2019).

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<sup>1</sup> Singular arcs appear when the switching function vanishes over a time interval.

## 2. SATURATION PHENOMENON

The purpose of this section is to recall some facts about minimum time control problems in the plane that will allow us to introduce the saturation phenomenon. Throughout the paper, the standard inner product in  $\mathbb{R}^2$  is written  $a \cdot b$  for  $a, b \in \mathbb{R}^2$ , and  $a^\perp$  denotes the vector  $a^\perp := (-a_2, a_1)$  orthogonal to  $a$ .

### 2.1 Pontryagin's Principle

We start by applying the classical optimality conditions provided by the Pontryagin Maximum Principle (PMP), see Pontryagin et al. (1964). Let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be two vector fields of class  $C^\infty$  and the system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)), \quad (1)$$

with admissible controls in the set

$$\mathcal{U} := \{u : [0, +\infty) \rightarrow [-1, 1] ; u \text{ meas.}\}.$$

Given an initial point  $x_0 \in \mathbb{R}^2$  and a non-empty closed subset  $\mathcal{T} \subset \mathbb{R}^2$ , we focus on the problem of driving (1) in minimal time from  $x_0$  to the target set  $\mathcal{T}$ :

$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t. } x_u(T_u) \in \mathcal{T}, \quad (2)$$

where  $x_u(\cdot)$  denotes the unique solution of (1) associated with the control  $u$  such that  $x_u(0) = x_0$ , and  $T_u \in [0, +\infty]$  is the first entry time of  $x_u(\cdot)$  into the target set  $\mathcal{T}$ . We suppose hereafter that optimal trajectories exist and we apply the PMP on (2). The Hamiltonian associated with (2) is the function  $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$H(x, p, p^0, u) := p \cdot f(x) + up \cdot g(x) + p^0.$$

If  $u$  is an optimal control and  $x_u$  is the associated trajectory steering  $x_0$  to the target set  $\mathcal{T}$  in time  $T_u \geq 0$ , the following conditions are fulfilled:

- There exist  $p^0 \leq 0$  and an absolutely continuous function  $p : [0, T_u] \rightarrow \mathbb{R}^2$  satisfying the adjoint equation

$$\dot{p}(t) = -\nabla_x H(x_u(t), p(t), p^0, u(t)) \quad \text{a.e. } t \in [0, T_u]. \quad (3)$$

- The pair  $(p^0, p(\cdot))$  is non-zero.
- The optimal control  $u$  satisfies the *Hamiltonian condition* almost everywhere over  $[0, T_u]$

$$u(t) \in \operatorname{argmax}_{\omega \in [-1, 1]} H(x_u(t), p(t), p^0, \omega). \quad (4)$$

- At the terminal time, the *transversality condition*<sup>2</sup>  $p(T_u) \in -N_{\mathcal{T}}(x_u(T_u))$  is fulfilled.

In the sequel, we only consider normal extremals that is, we take  $p^0 = -1$ , and we shall then write  $H(x, p, u)$  in place of  $H(x, p, p^0, u)$ . Since  $T_u$  is free and (1) is autonomous, the Hamiltonian  $H$  is zero along any extremal: for a.e.  $t \in [0, T_u]$ ,

$$H = p(t) \cdot f(x_u(t)) + u(t)p(t) \cdot g(x_u(t)) + p^0 = 0. \quad (5)$$

The *switching function*  $\phi$  is defined as

$$\phi(t) := p(t) \cdot g(x_u(t)), \quad t \in [0, T_u], \quad (6)$$

and it gives us (thanks to (4)) the following control law:

$$\begin{cases} \phi(t) > 0 \Rightarrow u(t) = +1, \\ \phi(t) < 0 \Rightarrow u(t) = -1. \end{cases} \quad (7)$$

A *switching time* is an instant  $t_c \in (0, T_u)$  such that the control  $u$  is discontinuous at time  $t_c$ . We say that the

<sup>2</sup>  $N_{\mathcal{T}}(x)$  is the limiting normal cone to  $\mathcal{T}$  at point  $x \in \mathcal{T}$ .

corresponding extremal trajectory has a *switching point* at time  $t_c$ . Of particular interest is the case when there is a time interval  $[t_1, t_2]$  such that the switching function vanishes over this interval, *i.e.*,

$$\phi(t) = p(t) \cdot g(x_u(t)) = 0, \quad t \in I.$$

By differentiating  $\phi$  twice w.r.t.  $t$ , one gets

$$\dot{\phi}(t) = p(t) \cdot [f, g](x_u(t)), \quad t \in [0, T_u],$$

$\ddot{\phi}(t) = p(t) \cdot [f, [f, g]](x_u(t)) + u(t)p(t) \cdot [g, [f, g]](x_u(t))$  almost everywhere over  $[0, T_u]$ . The notation  $[f, g](x)$  stands for the Lie bracket of  $f$  and  $g$  at point  $x$ . The *singular locus*  $\Delta_{SA}$  (in the state space) is defined as the (possibly empty) subset of  $\mathbb{R}^2$

$$\Delta_{SA} := \{x \in \mathbb{R}^2 ; \det(g(x), [f, g](x)) = 0\}. \quad (8)$$

For future reference, we set  $\delta_{SA}(x) := \det(g(x), [f, g](x))$  for  $x \in \mathbb{R}^2$ . Note that if an extremal is singular over a time interval  $[t_1, t_2]$ , then one has  $x_u(t) \in \Delta_{SA}$  for any  $t \in [t_1, t_2]$  because  $p(\cdot)$  must be non-zero and orthogonal to the vector space  $\operatorname{span}\{g(x_u(t)), [f, g](x_u(t))\}$  over  $[t_1, t_2]$ . The *singular control*  $u_s$  is then the value of the control for which the trajectory stays on the singular locus  $\Delta_{SA}$ . Supposing then that  $\phi = \dot{\phi} = 0$  over  $[t_1, t_2]$  gives:

$$u_s(t) := -\frac{p(t) \cdot [f, [f, g]](x_u(t))}{p(t) \cdot [g, [f, g]](x_u(t))}, \quad t \in [0, T_u], \quad (9)$$

provided that  $p(t) \cdot [g, [f, g]](x_u(t))$  is non zero for  $t \in [t_1, t_2]$ . This expression of the singular control does not guarantee that  $u_s$  is admissible, that is,  $u_s(t) \in [-1, 1]$ :

- When we have  $u_s(t) \in [-1, 1]$ , the point  $x_u(t)$  is said *hyperbolic* if  $p(t) \cdot [g, [f, g]](x_u(t)) > 0$ , and *elliptic* if  $p(t) \cdot [g, [f, g]](x_u(t)) < 0$  (see Bonnard and Pelletier (1995); Bonnard and Chyba (2003)).
- When we have  $|u_s(t)| > 1$  for some instant  $t$ , we say that a *saturation phenomenon* occurs and that the corresponding points of the singular locus are *parabolic* (see Bonnard and Pelletier (1995); Bonnard and Chyba (2003)).

Our purpose in what follows is precisely to investigate properties of the synthesis of optimal paths when saturation occurs.

### 2.2 Singular control and saturation phenomenon

We start by recalling classical expressions of the singular control that will allow us to properly define saturation points. The *collinearity set* associated with (1) is the (possibly empty) subset of  $\mathbb{R}^2$  defined as

$$\Delta_0 := \{x \in \mathbb{R}^2 ; \det(f(x), g(x)) = 0\}. \quad (10)$$

Define two functions  $\delta_0, \psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  as  $\delta_0(x) := \det(f(x), g(x))$ ,  $x \in \mathbb{R}^2$ , and

$$\psi(x) := -\frac{\det(g(x), [f, [f, g]](x))}{\det(g(x), [g, [f, g]](x))}, \quad x \in \mathbb{R}^2. \quad (11)$$

Using the PMP, we obtain the following expression of  $u_s$ .

*Lemma 1.* Suppose that  $\Delta_{SA} \neq \emptyset$ , that the mapping  $x \mapsto \det(g(x), [g, [f, g]](x))$  is non-zero over  $\Delta_{SA}$ , and consider a singular arc defined over an interval  $[t_1, t_2]$ . Then, one has:

$$u_s(t) = \psi(x(t)), \quad t \in [t_1, t_2], \quad (12)$$

where  $x(\cdot)$  is the corresponding singular trajectory such that  $x(t) \in \Delta_{SA}$  for  $t \in [t_1, t_2]$ .

**Proof.** The expression of  $u_s$  as a feedback control is classical (Boscain and Piccoli (2004), Bonnard and Chyba (2003)), see Bayen and Cots (2019) for more details.

To introduce the notion of saturation point, it is convenient to consider a parametrization of  $\Delta_{SA}$  as follows. When  $\Delta_{SA} \cap \Delta_0$  is non-empty,  $\Delta_{SA} \setminus \Delta_0$  can be divided into several subsets (called components hereafter):

$$\Delta_{SA} \setminus \Delta_0 = \bigcup_{k \in K} \gamma_k,$$

where  $K$  is an index set. By using the implicit function Theorem, we can show that each component of  $\Delta_{SA}$  can be parametrized by a one-to-one parametrization  $\zeta_k : J \rightarrow \gamma_k$ ,  $\tau \mapsto \zeta_k(\tau)$  of class  $C^1$  (where  $J$  is an interval of  $\mathbb{R}$ ), provided that the following condition is fulfilled

$$\forall x \in \Delta_{SA}, \det(g(x), [g, [f, g]](x)) \neq 0. \quad (13)$$

*Definition 2.* Given a component parametrized by  $\zeta$ , a point  $x^* := \zeta(\tau^*)$  with  $\tau^*$  in the interior of  $J$  is called *saturation point* if  $\psi(x^*) = 1$ ,  $\psi(\zeta(\tau)) \in (-1, 1)$  for any  $\tau \in J$  such that  $\tau < \tau^*$ , and  $\psi(\zeta(\tau)) > 1$  for any  $\tau \in J$  such that  $\tau > \tau^*$ . As well, we can define saturation points  $x^*$  such that  $\psi(x^*) = -1$ .

### 3. EXISTENCE OF A PRIOR-SATURATION POINT

In this section, we show that a prior-saturation phenomenon can occur whenever the system exhibits a saturation point. We start by introducing our main assumptions.

*Assumption 3.* The system (1) satisfies the following hypotheses:

- (i) One has  $\Delta_0 = \emptyset$  and  $\delta_0(x) < 0$  for all  $x \in \mathbb{R}^2$ .
- (ii) The set  $\Delta_{SA}$  is non-empty, simply connected, and has exactly one saturation point  $x^*$  with  $\psi(x^*) = 1$ .
- (iii) Along the singular locus, the strict (generalized) Legendre-Clebsch optimality condition is satisfied, that is, any singular extremal  $(x_u(\cdot), p(\cdot), u(\cdot))$  defined over  $[t_1, t_2]$  satisfies:

$$\frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial H}{\partial u}(x_u(t), p(t), u(t)) > 0, \quad \forall t \in [t_1, t_2]. \quad (14)$$

- (iv) If  $\Gamma_-$  is the forward semi-orbit of (1) with  $u = -1$  with the initial condition  $x^*$  at time 0, then

$$\mathcal{T} \cap \Gamma_- = \emptyset. \quad (15)$$

- (v) The target  $\mathcal{T}$  is reachable from every point  $x_0 \in \mathbb{R}^2$ .

*Remark 4.* (i) The hypothesis  $\Delta_0 = \emptyset$  is not restrictive since we could restrict our analysis to a component  $\gamma$  of  $\Delta_{SA}$  in place of  $\Delta_{SA}$ .

- (ii) By the previous computations, we can observe that (14) is equivalent to

$$\det(g(x), [g, [f, g]](x)) > 0, \quad \forall x \in \Delta_{SA},$$

and that, under the strict Legendre-Clebsch condition, the singular arc is a *turnpike*, Bonnard and Pelletier (1995).

The next proposition shows that an extremal trajectory containing a singular arc until the point  $x^*$  is not optimal.

*Proposition 5.* Suppose that Assumption 3 holds true, and consider an optimal trajectory steering  $x_0$  to the target  $\mathcal{T}$  in time  $T_u$ . Then, the corresponding extremal  $(x_u(\cdot), p(\cdot), u(\cdot))$  does not contain a singular arc defined over a time interval  $[t_1, t_2]$  such that  $x_u(t_2) = x^*$ .

**Proof.** We refer to Bayen and Cots (2019).

As an example, if  $x_0 := \zeta(\tau_0)$  belongs to the singular locus with  $\tau_0 < \tau^*$ , and if an optimal trajectory starting from  $x_0$  contains a singular arc, then the trajectory should leave the singular locus before reaching  $x^*$ . Let us insist on the fact that this property of leaving the singular locus before reaching  $x^*$  relies on the fact that the optimal trajectory should contain a singular arc. In the fed-batch model presented in Section 5, this property can be easily verified (see Bayen et al. (2015)).

We now introduce the following definition (in line with Ledzewicz and Schättler (2007); Schättler and Jankovic (1993); Schättler and Ledzewicz (2012)). Hereafter, the notation  $\mathcal{S}_{[\tau'_0, \tau_0]}$  denotes a singular arc passing through the points  $\zeta(\tau'_0)$  and  $\zeta(\tau_0)$  with  $\tau'_0 \leq \tau_0 < \tau^*$ .

*Definition 6.* Let  $\tau_0 < \tau^*$ . A point  $x_e := \zeta(\tau_e) \in \Delta_{SA}$  with  $\tau_0 < \tau_e < \tau^*$  is called a *prior-saturation point* if the singular arc  $\mathcal{S}_{[\tau_0, \tau]}$  ceases to be optimal for  $\tau \geq \tau_e$ .

This definition makes sense only for initial conditions  $\zeta(\tau_0)$  with  $\tau_0 < \tau^*$  because for  $\tau_0 \geq \tau^*$ , optimal controls are not singular (since the singular control is non-admissible). We highlight the dependency of  $x_e$  w.r.t. initial conditions  $\zeta(\tau_0) \in \Delta_{SA}$  as follows.

*Proposition 7.* Suppose that Assumption 3 holds true and that there are  $\tau_1, \tau_2 \in J$  with  $\tau_1 < \tau_2 < \tau^*$  such that any optimal trajectory starting from  $\zeta(\tau_0)$  with  $\tau_0 \in [\tau_1, \tau_2]$  contains a singular arc  $\mathcal{S}_{[\tau_0, \tau_2]}$ . Then, for any initial condition  $\tau_0 \in [\tau_1, \tau_2]$ , one has  $x_e = \zeta(\tau_e)$  with

$$\tau_e := \sup\{\tau \in J ; \mathcal{S}_{[\tau_1, \tau]} \text{ is optimal}\} \in [\tau_2, \tau^*). \quad (16)$$

Moreover, for any  $\tau_0 \in [\tau_e, \tau^*]$  an optimal trajectory starting at  $\zeta(\tau_0)$  leaves the singular locus at  $\zeta(\tau_0)$ .

**Proof.** We refer to Bayen and Cots (2019).

This property implies in particular that for every initial conditions  $x_0 := \zeta(\tau_0) \in \Delta_{SA}$  such that  $\tau_0 \in [\tau_1, \tau_2]$ , then the corresponding optimal path has a singular arc until the point  $x_e$  and a switching point at this point.

### 4. TANGENCY PROPERTY AND PRIOR-SATURATION PHENOMENON

The aim of this section is to prove the tangency property as stated in Theorem 15. In the example of Section 5, this property will hold at a certain *prior-saturation lift*:

*Definition 8.* Let  $x_e$  be a prior-saturation point. Any point  $z_e$  in the cotangent space at  $x_e$  is called a *prior-saturation lift* of  $x_e$ .

Classically, the PMP gives a set of nonlinear equations, the so-called *shooting equations*, that can be solved to compute the optimal extremal trajectory. We introduce some notation usually used to define this set of shooting equations, that we will be useful here to compute prior-saturation lifts; in a general frame, we can assume that the prior-saturation lifts involved in the optimal synthesis can be computed solving a set of nonlinear equations excerpted from the shooting equations.

We define the Hamiltonian lifts associated with  $f$  and  $g$  as

$$H_f(z) := p \cdot f(x); \quad H_g(z) := p \cdot g(x),$$

where  $z := (x, p)$  belongs to the cotangent bundle. All the others Hamiltonian lifts in the rest of the paper are defined like this. Define also the Hamiltonians  $H_{\pm} := H_f \pm H_g$  and  $H_s := H_f + u_s H_g$ , where  $u_s$  is viewed here as a function of  $z$ :

$$u_s(z) := -\frac{p \cdot [f, [f, g]](x)}{p \cdot [g, [f, g]](x)} = -\frac{H_{[f, [f, g]]}(z)}{H_{[g, [f, g]]}(z)}. \quad (17)$$

For any Hamiltonian  $H$  we define the *Hamiltonian system*  $\vec{H} := (\partial_p H, -\partial_x H)$ , and finally, we introduce the *exponential mapping*  $\exp(t\varphi)(z_0)$  as the solution at time  $t$  of the differential equation  $\dot{z}(s) = \varphi(z(s))$  with initial condition  $z(0) = z_0$ , where  $\varphi$  is supposed to be smooth.

Then, from a general point of view, we shall assume that prior-saturation lifts involved in the optimal synthesis will be given by solving a set of nonlinear equations of the following form:

$$F(t_b, z_b, \lambda) := \begin{pmatrix} H_{[f, g]}(\exp(-t_b \vec{H}_+)(z_b)) \\ G(t_b, z_b, \lambda) \end{pmatrix}, \quad (18)$$

where  $\lambda \in \mathbb{R}^k$  is a vector of  $k \in \mathbb{N}$  parameters, where  $F$  is a function from  $\mathbb{R}^{5+k}$  to  $\mathbb{R}^{5+k}$  and where  $G : \mathbb{R}^{5+k} \rightarrow \mathbb{R}^{4+k}$  is defined by

$$G(t_b, z_b, \lambda) := \begin{pmatrix} H_g(\exp(-t_b \vec{H}_+)(z_b)) \\ H_+(z_b) + p^0 \\ \Psi(z_b, \lambda) \end{pmatrix}, \quad (19)$$

with  $\Psi : \mathbb{R}^{4+k} \rightarrow \mathbb{R}^{2+k}$  a given function and  $p^0 = -1$  considering the normal case. We assume that all the functions  $F$ ,  $G$  and  $\Psi$  are smooth. It is important to notice that the mapping  $\Psi$  does not depend on  $t_b$  and that we can replace  $H_+$  by  $H_-$  without any loss of generality. We emphasize one more time that this form is very general.

Let  $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$  be a solution to the equation  $F = 0$  and define

$$z_e := \exp(-t_b^* \vec{H}_+)(z_b^*) \quad (20)$$

such that  $z_e \in \Sigma_{SA} := \{z \in \mathbb{R}^{2n} ; H_g(z) = H_{[f, g]}(z) = 0\}$ . We introduce the following assumptions at the point  $z_e$ .

*Assumption 9.* We have  $H_{[g, [f, g]]}(z_e) \neq 0$  and  $u_s(z_e) < 1$  with  $u_s$  the singular control given by (17).

*Assumption 10.* The matrix

$$\begin{bmatrix} \frac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \frac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix} \in \text{GL}_{4+k}(\mathbb{R}),$$

i.e., it is invertible in  $\mathbb{R}^{(4+k) \times (4+k)}$ .

*Remark 11.* Assumption 9 is related to the prior-saturation phenomenon while in combination with Assumption 10, it is related to the well-posedness of the shooting system  $F = 0$ . Besides, the point  $z_e$  is locally unique under these assumptions, according to the following result.

*Proposition 12.* Suppose that Assumptions 9 and 10 hold true. Then,

$$F'(t_b^*, z_b^*, \lambda^*) \in \text{GL}_{5+k}(\mathbb{R}).$$

**Proof.** Note that  $u_s(z_e) < 1$  plays a role in the result. We refer to Bayen and Cots (2019) for details.

*Lemma 13.* Suppose that Assumption 10 holds true. Then, there exists  $\varepsilon > 0$  and a  $C^1$ -map  $t_b \mapsto \sigma(t_b) := (z_b(t_b), \lambda(t_b)) \in \mathbb{R}^{4+k}$  defined over  $I_\varepsilon := (t_b^* - \varepsilon, t_b^* + \varepsilon)$ , that satisfies

$$\forall t_b \in I_\varepsilon, \quad G(t_b, \sigma(t_b)) = 0. \quad (21)$$

In addition, one has  $\sigma(t_b^*) = (z_b^*, \lambda^*)$  and  $\sigma'(t_b^*) = 0_{\mathbb{R}^{4+k}}$ .

**Proof.** This result is a simple application of the implicit function theorem. We refer to Bayen and Cots (2019) for details.

Let us introduce the mapping  $\varphi(t_b) := \exp(-t_b \vec{H}_+)(z_b(t_b))$  for  $t_b \in I_\varepsilon$  and define

$$\Sigma := \{\varphi(t_b) ; t_b \in I_\varepsilon\}. \quad (22)$$

*Remark 14.* The curve  $\Sigma$  is a switching curve in the cotangent bundle since one has  $H_g(\varphi(t_b)) = 0$  by definition of  $G$ . However, this switching curve is not necessarily optimal, that is, the optimal synthesis, with respect to the initial condition, may not contain  $\Sigma$ . Let us stratify  $\Sigma$  according to  $\Sigma = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$ , with

$$\begin{aligned} \Sigma_- &:= \{\varphi(t_b) ; t_b \in (t_b^* - \varepsilon, t_b^*)\}, \\ \Sigma_0 &:= \{\varphi(t_b^*)\} = \{z_e\}, \\ \Sigma_+ &:= \{\varphi(t_b) ; t_b \in (t_b^*, t_b^* + \varepsilon)\}. \end{aligned}$$

A typical situation is when  $\Sigma_- \cup \Sigma_0$  is contained in the optimal synthesis while  $\Sigma_+$  is not optimal for local and/or global optimality reasons. This is for instance the case in the example of Section 5.

Our first main result is given by Proposition 7 which states the existence of a prior-saturation point  $x_e$  in the state space under Assumption 3. Our second main result is the following.

*Theorem 15.* Suppose the existence of a triple  $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$  such that  $F(t_b^*, z_b^*, \lambda^*) = 0$ , with  $F$  defined by (18) and set  $z_e := \exp(-t_b^* \vec{H}_+)(z_b^*)$ . Suppose also that Assumption 10 holds true. Then, the switching curve  $\Sigma$  given by (22) is tangent at  $z_e$  to the forward semi-orbit  $\Gamma_+$  of  $\dot{z} = \vec{H}_+(z)$  starting from  $z_e$ .

**Proof.** This result is a consequence of Lemma 13. We refer to Bayen and Cots (2019) for details.

*Remark 16.* It is worth to mention that the tangency property is proved in the cotangent bundle, and thus it is also true in the state space at a prior saturation point (under the assumptions of Proposition 5).

*Remark 17.* In Bakir et al. (2019), the following semi-normal form called *bridge model*

$$\begin{cases} \dot{x}_1 = u(1 - x_2), \\ \dot{x}_2 = u(1 - x_1) + 1 - x_1^2 x_2, \end{cases}$$

is introduced. The associated singular locus is the union of two (simply connected) curves and has a singularity at their intersection. The authors solve the associated minimal time control problem from the initial condition  $(-1, -1)$  to the target  $(0, 0)$ , which exhibits a prior-saturation point and a bridge connecting the two singular loci. This generic model is a good illustration of the tangency property; indeed, for the time minimal synthesis the switching curve emanating from the prior-saturation point is tangent to the bridge. However, it is important to notice that the tangency property holds in a more general framework. We present in the following section an optimal synthesis which exhibits a prior-saturation point from where starts a bridge connecting the singular locus to the extended target set, and at which the tangency property holds.

## 5. ILLUSTRATION OF THE PRIOR-SATURATION PHENOMENON

A bioreactor operated in fed-batch is described by the controlled dynamics (see Moreno (1999)):

$$\begin{cases} \dot{s} = -\mu(s) \left( \frac{M}{v} + s_{in} - s \right) + \frac{Q_{max}(1+u)}{2v} (s_{in} - s), \\ \dot{v} = \frac{Q_{max}}{2} (1+u), \end{cases} \quad (23)$$

where  $s_{in}$  and  $s$  denote respectively the input substrate and substrate concentrations, and  $v$  is the volume of the reactor<sup>3</sup>. The parameter  $Q_{max} > 0$  is the maximal speed of the input pump (chosen large enough),  $\frac{Q_{max}}{2}(1+u)$  represents the input flow rate,  $u(\cdot)$  being the control variable with values in  $[-1, 1]$ , and  $M \in \mathbb{R}$  is a constant. The kinetics  $\mu$  of the reaction is of Haldane type

$$\mu(s) := \frac{\mu_h s}{K + s + \frac{s^2}{K_I}},$$

with a unique maximum  $s^* := \sqrt{K K_I} \in (0, s_{in})$  (parameters  $\mu_h, K, K_I$  are positive). This type of growth function models inhibition by substrate. It is worth mentioning that  $\mathcal{D} := (0, s_{in}] \times \mathbb{R}_+^*$  is invariant by (23). For waste water treatment purpose, the problem of interest is:

$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t.} \quad (s(T_u), v(T_u)) \in \mathcal{T}, \quad (24)$$

where  $\mathcal{T} := (0, s_{ref}] \times \{v_{max}\}$  is the target set,  $s_{ref} \ll s_{in}$  is a given threshold, and  $v_{max} > 0$  denotes the maximal volume of the bioreactor. For more details about this system, we refer to Moreno (1999); Bayen et al. (2015).

It appears that Problem (24) may exhibit a saturation phenomenon. Indeed, by using the PMP, we can check that there is a singular locus that is the line segment

$$\Delta_{SA} := \{s^*\} \times (0, v_{max}),$$

and that the singular control can be expressed as

$$u_s[v] := \frac{\mu(s^*) [M + v(s_{in} - s^*)]}{(s_{in} - s^*) Q_{max}} - 1,$$

(writing  $\dot{s} = 0$  along  $s = s^*$ ). It follows that there is a unique saturation point  $x_{sat} := (s^*, v^*)$  with  $v^* := \frac{2Q_{max}}{\mu(s^*)} - \frac{M}{s_{in} - s^*}$  and  $u_s[v^*] = 1$  if the following condition is fulfilled

$$0 < v^* < v_{max}. \quad (25)$$

This typically happens when  $v_{max}$  (the volume of water to be treated) is too large, see Bayen et al. (2015). Next, we suppose that (25) holds true.

At this step, we wish to know if prior-saturation occurs (according to Propositions 5 and 7). Doing so, let us check Assumption 3. One gets

$$\delta_0(s, v) = -\mu(s)(M/v + s_{in} - s)Q_{max}/2 < 0,$$

hence  $\Delta_0 \cap \mathcal{D} = \emptyset$  and  $\delta_0 < 0$  in  $\mathcal{D}$  (see Bayen and Cots (2019)). Now, the singular arc is of turnpike type and Legendre-Clebsch's optimality condition holds true because  $\mu$  has a unique maximum for  $s = s^*$ , see Bayen et al. (2013). In addition, observe that, in the  $(s, v)$ -plane, trajectories of (23) with  $u = -1$  are horizontal, hence, every arc with  $u = -1$  and starting at a volume value

<sup>3</sup> We chose to adopt the notation  $(s, v)$  in place of  $(x_1, x_2)$  as in the bioprocesses literature for fed-batch operations.

$v_0 < v_{max}$  never reaches the target set  $\mathcal{T}$ . Finally,  $\mathcal{T}$  is reachable from  $\mathcal{D}$  taking the control  $u = +1$  until reaching  $v = v_{max}$  and then  $u = -1$  until reaching  $s_{ref}$ .

Second, let us verify the hypotheses of Proposition 7. Doing so, let  $v \mapsto \hat{s}(v)$  be the unique solution to the Cauchy problem

$$\begin{aligned} \frac{ds}{dv} &= - \left( -\frac{\mu(s)}{Q_{max}} \left[ \frac{M}{v} + s_{in} - s \right] + \frac{s_{in} - s}{v} \right), \\ s(v_{max}) &= s^*, \end{aligned}$$

(the solution of (23) with  $u = 1$  backward in time from  $(s^*, v_{max})$ ). From Bayen et al. (2015), if there exists  $v_* \in (0, v^*)$  such that  $\hat{s}(v_*) = s^*$ , then optimal paths starting at a volume value sufficiently small necessarily contain a singular arc (this actually follows using the PMP). Now, by using Cauchy-Lipschitz's Theorem, the existence of  $v_*$  is easy to verify when  $M = 0$ , and thus, it is also verified for small values of the parameter  $M$  (by a continuity argumentation). To pursue our analysis, we suppose next the existence of  $v_* \in (0, v^*)$ . We are then in a position to apply Propositions 5 and 7. It follows that there is a unique volume value  $v_e \in (0, v^*)$  such that any singular arc starting at a volume value  $v_0 < v_e$  will be optimal only until  $v_e$ . In addition, combining this result with a study of extremals using the PMP, we obtain that

- if the initial condition is  $(s^*, v_0)$  with  $v_0 < v_e$ , then the optimal path is of the form  $\sigma_s \sigma_+^b \sigma_-$  (see below for the definition of  $\sigma_+^b$ );
- if the initial condition is  $(s^*, v_0)$  with  $v_0 \geq v_e$ , then the optimal path is of the form  $\sigma_+ \sigma_-$ ;
- for any initial condition  $(s_{in}, v_0)$  with  $v_e \leq v_0 < v_{max}$ , the optimal path is of the form  $\sigma_- \sigma_+ \sigma_-$  where the first switching time appears on a switching curve emanating from  $(s^*, v_e)$ .

To determine the prior-saturation point  $x_e := (s^*, v_e)$  numerically, we proceed as in Section 4. For this application model, it is convenient to introduce an *extended target set* as  $\overline{\mathcal{T}} := (0, s_{in}] \times \{v_{max}\}$  (observe that for initial conditions on  $\overline{\mathcal{T}}$ , optimal paths are  $\sigma_-$  arcs). In this context, a *bridge* is defined as an arc  $\sigma_+$  (denoted by  $\sigma_+^b$ ) on  $[0, t_b]$  such that

$$\phi(0) = \dot{\phi}(0) = \phi(t_b) = 0 \quad \text{and} \quad v(t_b) = v_{max},$$

where  $\phi$  is the switching function defined by (6) and  $t_b$  is the time to steer  $x_e$  at time 0 to the extended target set  $\overline{\mathcal{T}}$  with  $u = +1$ . To compute  $x_e$ , we need to compute the extremities of the bridge together with its length. Denoting by  $t_b^*$  the length of the bridge and by  $z_b^*$  its extremity in the cotangent bundle whose projection on the state space belongs to  $\overline{\mathcal{T}}$ , the point  $(t_b^*, z_b^*)$  is then a solution of the equation  $F_{bio} = 0$  with

$$F_{bio}(t_b, z_b) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_g(\exp(-t_b \overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ H_g(z_b) \\ v_b - v_{max} \end{pmatrix}, \quad (26)$$

where  $(s_b, v_b)$  is the projection of  $z_b$  on the state space and vector fields  $f, g$  are given by (23). From Theorem 15, the bridge is then tangent to the switching curve at  $x_e$  (the projection of  $\Sigma$  given by (22) onto the state space). To conclude this part, let us comment Fig. 1 on which

the optimal synthesis is plotted in a neighborhood of the prior-saturation point:

- In black, the switching curve  $\Sigma^\pi$  emanates from the prior-saturation point. It is computed using the shooting functions  $F_{bio}$ .
- The synthesis is such that trajectories are horizontal ( $u = -1$ ) until reaching  $\Delta_{SA}$  or the switching curve. For initial conditions with a substrate concentration less than  $s^*$  and  $v_0 \geq v_e$ , then  $u = 1$  is optimal until reaching  $\bar{\mathcal{T}}$ .

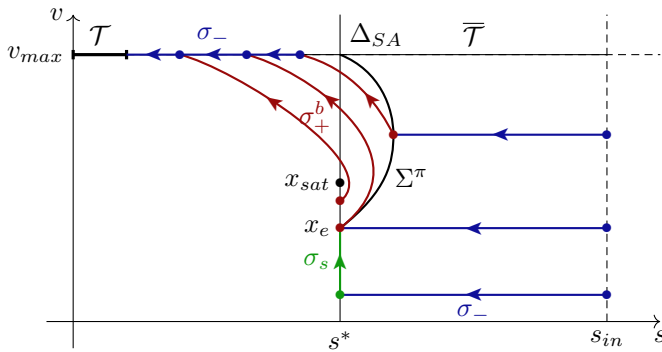


Fig. 1. Minimal time synthesis for (24): the target set  $\mathcal{T} = (0, s_{ref}] \times \{v_{max}\}$  is in black (left). The switching curve  $\Sigma^\pi$  (in black) is tangent to the bridge  $\sigma_+^b$  (in red) at  $x_e$ . Arcs with  $u = +1$  (resp.  $u = -1$ ) are depicted in red (resp. in blue).

## 6. CONCLUSION

Even though the tangency property between the bridge and the switching curve provides useful informations on the minimum time synthesis when prior saturation occurs (typically, under assumptions of Proposition 5), it remains valid in a larger context (under the hypotheses of Theorem 15) and not only in the framework of saturation and prior-saturation of the singular control for affine-control systems in the plane. It also appears in other settings such as in Lagrange control problems, see, *e.g.*, Kalboussi et al. (2019). Future works could then investigate prior-saturation phenomenon and the tangency property in other frameworks or in dimension  $n \geq 3$ .

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