A new proof of optimality conditions for the time of crisis via regularization

Térence Bayen, Kenza Boumaza, Alain Rapaport

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Abstract

Our aim in this paper is to derive optimality conditions for the time of crisis problem under a weaker hypothesis than the usual one encountered in the hybrid setting, and which asserts that any optimal solution should cross the boundary of the constraint set transversally. Doing so, we apply the Pontryagin Maximum Principle to a sequence of regular optimal control problems whose integral cost approximates the time of crisis. Optimality conditions are derived by passing to the limit in the Hamiltonian system. This convergence result essentially relies on the boundedness of the sequence of adjoint vectors in L^∞. Our main contribution is to relate this property to the boundedness in L^1 of a suitable sequence which allows to bypass the use of the transverse hypothesis on optimal paths.

1 Introduction

This paper proposes a new approach to derive optimality conditions for the so-called *time of crisis problem* [5] as well as (new) sufficient conditions ensuring the well-posedness of this approach. Such conditions will slightly differ of those available in the literature that involve the behavior of an (a priori unknown) optimal path.

Originally, the time of crisis problem was introduced in [15], and it consists in minimizing the total time spent by a solution of a control system outside a given constraint set. It is of particular interest whenever it is not possible to maintain the system in such a set. In that case, alternative strategies consist in finding a control policy such that the associated solution spends the minimum of time outside the constraint set. The time of crisis arises in the context of viability theory [2, 3], see, *e.g.*, a case study in ecology in [7], and more generally whenever one is unable to maintain a controlled dynamics within a prescribed constraint set over a time windows.

From a theoretical point of view, the formulation of the time of crisis involves a discontinuous function w.r.t. the state, namely the indicator function of the constraint set. The integrand is then equal to 0 or 1 depending on the position of the system in the state space. Therefore, the *Pontryagin Maximum Principle* (PMP), see [19], cannot be applied straightforwardly to derive necessary conditions. Nevertheless, various approaches have been proposed in the literature to study this issue and we now wish to give an overview of the available methods in order to highlight the differences with our approach.

In the first paper about the time of crisis (see [15]), the optimal control problem was tackled via a dynamic programming approach. The question of necessary optimality conditions has been investigated more recently in [5] using the so-called *hybrid maximum principle* (HMP) which is an extension of the PMP adapted to hybrid systems (see [16, 17, 10]). In [5], the authors provide necessary conditions by a direct application of this principle that requires a so-called *transverse hypothesis* on optimal trajectories which is as follows: any optimal solution crosses the boundary of the constraint set transversally. As in [17], this hypothesis is crucial for the obtention of necessary conditions (in particular, for the jump of the covector). Thanks to this hypothesis, it is also shown in [5, 6] that extremals of a regularized[1] optimal control problem converge, up to a sub-sequence, to an extremal of the time of crisis problem. The methodology is in line with [17]. Note that first and second order conditions have also been derived in [8] using the PMP on a reformulation of the time of crisis problem obtained via an augmentation technique (in the spirit of [12, 13, 14]). Let us also point out the paper [1] in which an approximation technique

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[1]In [5, 7], the regularization technique is based on the Moreau-Yosida approximation of the indicator function.

is introduced in order to obtain necessary conditions. The study of convergence of regularized extremals relies on a similar transverse hypothesis on an optimal solution as in the application of the HMP. In addition, the approximated optimal control problem involves the (desired) optimal solution. Let us point out that this approach is of interest for obtaining optimality conditions via the use of the PMP on a smooth problem, however, it is not usable from a numerical point of view since the sequence of approximated problems itself involves the optimal solution.

Let us now describe more into details the content of this paper. As we have seen, the time of crisis can be viewed as an application of the HMP on a discontinuous problem w.r.t. the state. The HMP available in the literature is a powerful tool but its application requires an optimal solution to satisfy a transverse assumption related to the boundary of the constraint set (see [17]). In practice, this condition is hardly possible to check if one does not know in advance the optimal solution or estimates it with enough accuracy. That is why, we ask whether or not it is possible to derive optimality conditions for discontinuous integrands w.r.t the state without the use of this hypothesis. Doing so, we introduce a sequence of regular optimal control problems whose integral cost approximates the time of crisis (this is made possible using mollifiers). Our contribution is twofold:

- We propose in the context of the time of crisis problem a sufficient condition for the obtention of necessary optimality conditions. This condition relies on the data of the problem and on a sequence of approximated solutions, and it can be easily checked.
- Necessary conditions for the time of crisis are derived under this condition which also covers the transverse case.

As a byproduct of our approach (and in contrast with [1, 20] for instance), the sequence of approximated optimal control problems allows to approach an optimal solution of the original problem with a solution associated with a regular optimal control problem, that can be solved numerically with existing efficient methods.

The paper is organized as follows. In Section 2, we introduce the time of crisis and the regularization scheme, and we also apply the PMP on the regularized optimal control problem. In section 3, we study properties of a sequence of approximated solutions (namely, the integral of the approximated solution computed along the mollifier). This sequence will be crucial to introduce an auxiliary hypothesis in Section 5 allowing then the derivation of optimality conditions for the time of crisis. Section 4 provides optimality conditions for the time of crisis under the hypothesis that the suitable sequence is bounded (and so, without a transverse hypothesis which is the novelty here). Finally, a sufficient condition involving this sequence is presented in Section 5 and the last section provides some examples that highlight the convergence of the sequence of adjoint vectors to a discontinuous covector having a jump at each crossing time. The paper ends up with a conclusion.

2 Definitions and regularization of the time crisis problem

2.1 Notations and main hypotheses

Throughout the paper, $m, n \geq 1$ are integers and $T > 0$. Let us introduce the following notations:

- For $x, y \in \mathbb{R}^n$, $|x|$ and $x \cdot y$, denote the euclidean norm of x and the scalar product between x and y. If A is a square matrix of dimension n, $\|A\|$ stands for the norm of a A and its transpose is written $A^\top$.
- Given a mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, we denote respectively by $D_x f(x, u)$, $D_u f(x, u)$ the Jacobian matrix of f w.r.t. variables x and u at some point $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. The notations $\nabla\varphi(x)$, $D_{xx}\varphi(x)$ denote the gradient and the Hessian of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at some point $x \in \mathbb{R}^n$.
- Given two integers $k, p \geq 1$ and a function $w : [0, T] \to \mathbb{R}^k$, the notation $\|w\|_{L^p(I\,;\mathbb{R}^k)}$ will stand for the L^p-norm of w over some time interval $I \subset [0, T]$.
- If $g \in L^\infty([0, T]\,; \mathbb{R}^s)$, $s \in \mathbb{N}^*$, we denote by $g(t^\pm)$ the right and left limits (when it exists) of g at point t. In the same way, we shall denote by $\dot{g}(t^\pm)$ the right and left derivative of a scalar function g (when it exists).

In the sequel, we consider two non-empty subsets U and K of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, as well as two mappings $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (the dynamics). Throughout the paper, we suppose that the following hypotheses are satisfied:

(H1) The set U is compact and f is of class C^1 with linear growth, *i.e.*, there is $c \geq 0$ such that for every $(x, u) \in \mathbb{R}^n \times U$, one has $|f(x, u)| \leq c(|x| + 1)$.

(H2) For every $(x,p) \in \mathbb{R}^{2n}$, the set

$$\bigcup_{u \in U} \begin{bmatrix} f(x,u) \\ -D_x f(x,u)^\top p \end{bmatrix}$$

is a non-empty compact[2] convex subset of $\mathbb{R}^{2n}$.

(H3) We suppose that φ is of class C^2, that the set K is with non-empty interior and is the 0-sub-level set of φ:

$$K = \{x \in \mathbb{R}^n \ ; \ \varphi(x) \le 0\}.$$

(H4) For every $x \in \partial K$ (the boundary of K), one has $\varphi(x) = 0$ and $\nabla\varphi(x) \neq 0$.

Note that (H2) is fulfilled whenever the dynamics f is affine w.r.t. to the control u.

2.2 The time of crisis

Throughout the paper, we consider the admissible control set

$$\mathcal{U} := \{u : [0,T] \to U \ ; \ u \in L^\infty([0,T] \, ; U)\}.$$

Given an initial condition $x_0 \in \mathbb{R}^n$, the *minimal time crisis* (over $[0,T]$) is defined as

$$\inf_{u \in \mathcal{U}} \int_0^T \mathbb{1}_{K^c}(x_u(t)) \, \mathrm{d}t, \tag{2.1}$$

where $\mathbb{1}_{K^c}$ denotes the characteristic function of the complement of K, *i.e.*, $\mathbb{1}_{K^c}(x) = 1$ if $x \notin K$, $\mathbb{1}_{K^c}(x) = 0$ if $x \in K$, and $x_u(\cdot)$ is the unique (global) solution to the Cauchy problem

$$\left\{ \begin{array}{ll} \dot{x}(t) & = f(x(t), u(t)) \quad \text{a.e. } t \in [0,T], \\ x(0) & = x_0. \end{array} \right. \tag{2.2}$$

Under the previous assumptions, the existence of an optimal control for (2.1) follows easily from the upper semi-continuity of $\mathbb{1}_{K^c}$ (see, *e.g.*, [5, Proposition 2.1] for more details). An important feature for studying optimality conditions of (2.1) is the notion of crossing time that we recall below.

Definition 2.1. *Given a solution $x(\cdot)$ of (2.2), let us define the absolutely function ρ as:*

$$\rho(t) := \varphi(x(t)), \quad t \in [0,T]. \tag{2.3}$$

(i) *A crossing time from K to K^c, resp. from K^c to K, is a time $\tau \in (0,T)$ for which there is $\eta > 0$ with $[\tau - \eta, \tau + \eta] \subset [0,T]$ such that $x(t) \in K$, resp. $x(t) \in K^c$, for $t \in [\tau - \eta, \tau]$ and $x(t) \notin K$, resp. $x(t) \in K$, for $t \in (\tau, \tau + \eta]$. We shall say that a crossing time is "outward" if x crosses ∂K from K to K^c, and "inward" if it crosses from K^c to K.*

(ii) *A crossing time τ is transverse if moreover the function ρ admits non-null left and right limits, i.e.,*

$$\dot{\rho}(\tau^\pm) = \lim_{t \to \tau^\pm} \nabla\varphi(x(t)) \cdot \dot{x}(t) \neq 0 \tag{2.4}$$

(negative for an outward crossing time, positive for an inward crossing time.)

Remark 2.1. *Definition 2.1 (i) is equivalent to say that τ is an isolated root of ρ such that the map $t \mapsto \rho(t)(t-\tau)$ is locally of constant sign (positive from from K to K^c, negative from from K^c to K).*

Throughout the paper, we suppose that an optimal solution x^* of (2.1) has a finite number $r \ge 1$ of (alternated) crossing times $(\tau_i)_{1 \le i \le r}$ such that

$$0 < \tau_1 < \tau_2 < \cdots < \tau_r < T.$$

In particular, we will not consider trajectories that may cross the boundary of K an infinite number of times over $[0,T]$ (such as chattering [22]).

[2] The compactness property actually follows from the continuity of f and $D_x f$ and the compactness of U.

2.3 Regularization scheme

The approach developed in the present paper is an approximation procedure of Problem 2.1 with a sequence of regular problems that can be solved with standard optimality conditions (such as the PMP) or existing numerical tools. It will allow us to recover the conditions obtained for instance in [5] using the HMP [17]. As mentioned in the introduction, other authors considered regularization of problems similar to (2.1), but requiring an a priori knowledge of an optimal control [1], which therefore cannot be used as a practical approximation, in contrast with our approach. In addition, we shall see that the sufficient condition for the derivation of optimality conditions that we obtain in Section 5 does not involve the assumption that each crossing time of an optimal solution is transverse, as it is required in the HMP. Instead, this condition relies only on the boundedness in L^1 of a suitable sequence related to the regularized problem, that can be tested numerically.

Let us now introduce a regularized scheme associated with (2.1). Doing so, we consider a sequence $G_n : \mathbb{R} \to \mathbb{R}$ approximating the Heaviside step function[3] G as follows:

- G_n is of class C^1 and converges pointwise to G: for every $\sigma \in \mathbb{R}$, $G_n(\sigma) \to G(\sigma)$ when $n \to +\infty$.
- there are two sequences of real numbers $(a_n)_n$, $(b_n)_n$ such that for every $n \in \mathbb{N}$, one has $a_n \leq 0$, $b_n \geq 0$, and $G_n(\sigma) = 0$, resp. $G_n(\sigma) = 1$ for every $\sigma \leq a_n$, resp. $\sigma \geq b_n$. In addition, $a_n \uparrow 0$ and $b_n \downarrow 0$ when $n \to +\infty$.

In view of its definition, the function G_n is Lipschitz continuous, and its Lipschitz constant L_n is such that $L_n \to +\infty$ whenever $n \to +\infty$. Note also that

$$h_n := G_n',$$

is a mollifier, *i.e.*, for every $n \in \mathbb{N}$, its support is contained in $[a_n, b_n]$ and $\int_{\mathbb{R}} h_n(\sigma)\mathrm{d}\sigma = 1$. In addition, one has $\sup_{\sigma\in\mathbb{R}} |h_n'(\sigma)| \to +\infty$ whenever $n \to +\infty$.

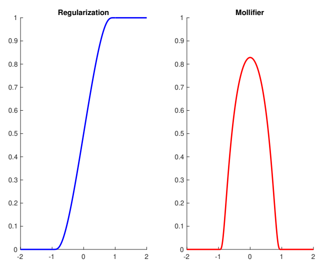

Figure 1: Plot of G_n (Fig. left) and of its derivative h_n (Fig. right) for $n = 1$.

In the sequel, we consider the following sequence of optimal control problems

$$\inf_{u\in\mathcal{U}} \int_0^T G_n(\varphi(x_u(t)))\,\mathrm{d}t, \tag{2.5}$$

where $x_u(\cdot)$ is the unique solution to (2.2). The existence of an optimal control of (2.5) is straightforward using Fillipov's existence Theorem [21] under our assumptions. Hereafter, we denote by (x_n, u_n) an optimal pair of (2.5). Following [9], we can show that, up to a subsequence, the sequence x_n converges strongly-weakly[4] to an optimal pair (x^*, u^*) of (2.1). Let us stress that (u_n) may not converge pointwise to u^*.

2.4 Optimality conditions for the regularized problem

We are now in a position to apply the Pontryagin Maximum Principle on Problem (2.5). Let $H_n : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be the Hamiltonian[5] associated with (2.5) defined as:

$$H_n(x, p, u) = p \cdot f(x, u) - G_n(\varphi(x)).$$

[3] We define G as the function such that $G(\sigma) = 0$, resp. $G(\sigma) = 1$ whenever $\sigma \leq 0$, resp. $\sigma > 0$.

[4] This means that $(x_n)_n$ uniformly converges to x^* over $[0, T]$ and that $(\dot{x}_n)_n$ weakly converges to $\dot{x}^*$ in $L^2([0, T]\,;\,\mathbb{R}^n)$.

[5] Since no terminal constraint appear in (2.5), one can directly take $p^0 = -1$ for the multiplier associated to the objective function G_n, *i.e.*, only normal extremals occur.

Since (x_n, u_n) is optimal for (2.5), there is an absolutely continuous map $p_n : [0,T] \to \mathbb{R}^n$ satisfying the adjoint equation $\dot{p}_n = -\frac{\partial H}{\partial x}$ almost everywhere, that is,

$$\left| \begin{array}{ll} \dot{p_n}(t) & = -D_x f(x_n(t), u_n(t))^\top p_n(t) + h_n(\varphi(x_n(t)))\nabla\varphi(x_n(t)) \quad \text{a.e. } t \in [0,T], \\ p_n(T) & = 0, \end{array} \right. \tag{2.6}$$

as well as the Hamiltonian condition which can be written

$$\forall u \in U, \quad p_n(t) \cdot f(x_n(t), u) \leq p_n(t) \cdot f(x_n(t), u_n(t)) \quad \text{a.e. } t \in [0,T]. \tag{2.7}$$

A triple (x_n, p_n, u_n) satisfying (2.2)-(2.6)-(2.7) is called an extremal (recall that only normal extremals occur here as there is no terminal condition). Let us observe that the problem is autonomous, therefore, the Hamiltonian is conserved along any extremal. For every $n \in \mathbb{N}$, there is $\tilde{H}_n \in \mathbb{R}$ such that:

$$\tilde{H}_n = H_n(x_n(t), p_n(t), u_n(t)) = p_n(t) \cdot f(x_n(t), u_n(t)) - G_n(\varphi(x_n(t))) \quad \text{a.e. } t \in [0,T].$$

Remark 2.2. *Since $(x_n)_n$ strongly-weakly converges to x^* which satisfies* (H′)*, it can be observed that $t \mapsto h_n(\varphi(x_n(t)))$ takes arbitrarily large values in* (2.6)*. Hence, we can expect the sequence $(\dot{p}_n)_n$ to be unbounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. We shall see, that under* (H′) *(that is whenever every crossing time of x^* is transverse), $(p_n)_n$ is indeed bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ even though this is not the case for $(\dot{p}_n)_n$. This is actually the main difficulty for studying the behavior of* (2.6) *and for passing to the limit when $n \to +\infty$.*

The boundedness of the sequence $(p_n)_n$ is related to the behavior of the sequence $(I_n)_n$ defined as

$$I_n := \int_0^T h_n(\varphi(x_n(t)))\,\mathrm{d}t. \tag{2.8}$$

As this integral will play a crucial role in the establishment of optimality conditions, passing to the limit into (2.6) when $n \to +\infty$, we devote the next session to analysis of its properties.

3 Properties of the sequence of integrals $(I_n)_n$

We start by proving that $(I_n)_n$ is bounded if and only if $(p_n)_n$ is bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. Recall that the limiting path x^* has a finite number of crossing times $(\tau_i)_{1\leq i\leq r}$. Set

$$\delta := \min_{0\leq i\leq r} (\tau_{i+1} - \tau_i),$$

with the convention $\tau_0 := 0$, $\tau_{r+1} := T$, and define the sub-sets

$$\mathcal{I}_\eta := \bigcup_{1\leq i\leq r} [\tau_i - \eta, \tau_i + \eta]; \quad \mathcal{J}_\eta := [0,T] \setminus \mathcal{I}_\eta.$$

The following property will be used at several places.

Property 3.1. *For all $\eta \in (0,\delta)$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ and $t \in \mathcal{J}_\eta$, one has $h_n(\varphi(x_n(t))) = 0$.*

Proof. Take $\eta \in (0,\delta)$. Since $(x_n)_n$ uniformly converges to x^* and as $\varphi(x^*(t)) = 0$ if and only if $t \in \{\tau_1, ..., \tau_r\}$, there are $N_1 \in \mathbb{N}$ and $\gamma > 0$ such that

$$\forall n \geq N_1,\ \forall t \in J_\eta, \quad |\varphi(x_n(t))| \geq \gamma.$$

Now, recall that both sequences $(a_n)_n$, $(b_n)_n$ defining the the support of h_n converge to zero, whence the result. □

Next, we have the following equivalence between the boundedness of $(p_n)_n$ and $(I_n)_n$.

Proposition 3.1. *The sequence $(p_n)_n$ is bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ if and only if $(I_n)_n$ is bounded in $\mathbb{R}_+$.*

Proof. Since $p_n(T) = 0$ for every $n \in \mathbb{N}$, the boundedness of $(p_n)_n$ easily follows from the boundedness of $(I_n)_n$ using Gronwall's Lemma and the fact that $(x_n)_n$ and $(u_n)_n$ are bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ and $L^\infty([0,T]\,;\mathbb{R}^m)$ respectively.

Let us now assume that $(p_n)_n$ is bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. By (2.6) one has

$$\int_0^T h_n(\varphi(x_n(t)))|\nabla\varphi(x_n(t))|^2 \mathrm{d}t = \int_0^T \dot{p}_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t + \int_0^T D_x f(x_n(t),u_n(t))^\top p_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t. \tag{3.9}$$

From (H4) and using the uniform convergence of $(x_n)_n$ to x^*, there are $\eta \in (0,\delta)$, $N_1 \in \mathbb{N}$, and $\gamma' > 0$ such that

$$\forall t \in \mathcal{I}_\eta,\ \forall n \geq N_1, \quad |\nabla\varphi(x_n(t))|^2 \geq \gamma'.$$

From property 3.1, there is $N \in \mathbb{N}$ such that for every $n \geq N$ and every $t \in \mathcal{J}_\eta$, one has $h_n(\varphi(x_n(t))) = 0$. It follows that for every $n \geq N_2 := \max(N, N_1)$, one has:

$$\int_0^T h_n(\varphi(x_n(t)))|\nabla\varphi(x_n(t))|^2\,\mathrm{d}t \geq \gamma' \int_{\mathcal{I}_\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t. \tag{3.10}$$

Now, from the hypotheses on f and φ, there is $C \geq 0$ such that:

$$\forall n \in \mathbb{N}, \quad \forall t \in [0,T], \quad |D_x f(x_n(t),u_n(t))^\top p_n(t)\cdot\nabla\varphi(x_n(t))| \leq C.$$

By an integration by parts, we obtain using the terminal condition $p_n(T) = 0$ for every $n \in \mathbb{N}$:

$$\int_0^T \dot{p}_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t = -p_n(0)\nabla\varphi(x_0) - \int_0^T D_{xx}\varphi(x_n(t))p_n(t)\cdot\dot{x}_n(t)\,\mathrm{d}t.$$

As the sequence $(x_n)_n$ is uniformly bounded, we deduce that there is $C' \geq 0$ such that for every $n \in \mathbb{N}$, one has

$$\left|\int_0^T \dot{p}_n(t)\cdot\nabla\varphi(x_n(t))\,\mathrm{d}t\right| \leq |p_n(0)||\nabla\varphi(x_0)| + \int_0^T \|D_{xx}\varphi(x_n(t))\|\,|p_n(t)|\,|\dot{x}_n(t)|\,\mathrm{d}t \leq C' \tag{3.11}$$

Combining (3.9)-(3.10)-(3.11) then implies

$$\forall n \geq N_2, \quad \gamma' \int_{\mathcal{I}_\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t \leq CT + C'.$$

We have thus proved that the sequence $\left(\int_{\mathcal{I}_\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t\right)_n$ is bounded. Since $h_n(\varphi(x_n(t))) = 0$ for every $t \in \mathcal{J}_\eta$ and every $n \geq N$, the result then follows. □

This proposition will be used later on to obtain optimality conditions for (2.1) by letting $n \to +\infty$ in (2.6) and by reasoning by contradiction supposing that $(p_n)_n$ is unbounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. We now aim at studying convergence properties of the sequence $(I_n)_n$. We show that the sequence $(I_n)_n$ enjoys a kernel-type property. For any $1 \leq i \leq r$, define the partial integrals

$$\tilde{I}_{i,n}(\eta) := \int_{\tau_i-\eta}^{\tau_i+\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t, \quad \eta \in (0,\delta).$$

Lemma 3.1. *Suppose that $(I_n)_n$ is bounded. Then, for any $i \in \{1,...,r\}$, there exists $\ell_i \in \mathbb{R}_+$ such that for every $\eta \in (0,\delta]$, $\tilde{I}_{i,n}(\eta) \to \ell_i$ whenever $n \to +\infty$.*

Proof. Since $(I_n)_n$ is bounded so is $(\tilde{I}_{i,n}(\eta))_n$ for $\eta \in (0,\delta]$, hence, we may assume that there exists $\ell_i \in \mathbb{R}_+$ such that (up to a sub-sequence), $\tilde{I}_{i,n}(\delta) \to \ell_i$ when $n \to +\infty$. We can then write

$$\tilde{I}_{i,n}(\delta) - \tilde{I}_{i,n}(\eta) = \int_{\tau_i-\delta}^{\tau_i-\eta} h_n(\varphi(x_n(t)))\,\mathrm{d}t + \int_{\tau_i+\eta}^{\tau_i+\delta} h_n(\varphi(x_n(t)))\,\mathrm{d}t.$$

Let then

$$\gamma_\eta := \min_{t\in[\tau_i-\delta,\tau_i-\eta]\cup[\tau_i+\eta,\tau_i+\delta]} |\varphi(x^*(t))| > 0.$$

Recall that $(x_n)_n$ uniformly converges to x^* when $n \to +\infty$. Thus, there exists $N \in \mathbb{N}$ such that:

$$\forall n \geq N,\ \forall t \in [\tau_i - \delta, \tau_i - \eta] \cup [\tau_i + \eta, \tau_i + \delta], \quad |\varphi(x_n(t))| \geq \frac{\gamma_\eta}{2}.$$

Now, both sequence $(a_n)_n$ and $(b_n)_n$ converge to zero, hence there exists $N' \geq N$ such that

$$\forall n \geq N',\ [a_n, b_n] \subset \left[-\frac{\gamma_\eta}{2}, \frac{\gamma_\eta}{2}\right].$$

Because the support of h_n is contained in $[a_n, b_n]$, we conclude that

$$\forall n \geq N',\ \tilde{I}_{i,n}(\delta) = \tilde{I}_{i,n}(\eta).$$

This proves that $\tilde{I}_{i,n}(\eta) \to \ell_i$ when $n \to +\infty$. Since $\eta \in (0, \delta]$ is arbitrary, the result follows. $\square$

Thanks to this lemma, we can now show the following result which provides a kernel-type property[6] of (I_n) and that will be crucial hereafter to study the convergence of $(p_n)_n$.

Proposition 3.2. *If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a function of class C^1 and $(I_n)_n$ is bounded, then:*

$$\forall \varepsilon > 0, \exists \eta_i \in (0, \delta], \exists N \in \mathbb{N},\ \forall n \geq N, \qquad \left| \int_{\tau_i - \eta_i}^{\tau_i + \eta_i} h_n(\varphi(x_n(t))) g(x_n(t))\, \mathrm{d}t - \ell_i g(x^*(\tau_i)) \right| \leq \varepsilon. \tag{3.12}$$

In addition, η_i goes to zero as $\varepsilon \downarrow 0$.

Proof. For $\eta \in (0, \delta]$, one can write:

$$\int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t))) g(x_n(t))\, \mathrm{d}t - \ell_i g(x^*(\tau_i)) = \underbrace{\int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t)))[g(x_n(t)) - g(x^*(t))]\, \mathrm{d}t}_{\Lambda_n^1(\eta)}$$
$$+ \underbrace{\int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t)))[g(x^*(t)) - g(x^*(\tau_i))]\, \mathrm{d}t}_{\Lambda_n^2(\eta)} + (\tilde{I}_{i,n}(\eta) - \ell_i) g(x^*(\tau_i)).$$

Let then $\varepsilon > 0$ be fixed and set $M := \sup_n I_n$. By continuity of x^* at $t = \tau_i$, there exists $\eta_i > 0$ such that

$$|g(x^*(t)) - g(x^*(\tau_i))| \leq \frac{\varepsilon}{3M}, \quad t \in [\tau_i - \eta_i, \tau_i + \eta_i]. \tag{3.13}$$

Without any loss of generality, one can choose η_i such that it goes to zero as ε tends to 0 because $t \mapsto (g \circ x^*)(t)$ is Lipschitz continuous over $[0, T]$ (since g is of class C^1 and $\dot{x}^*$ is bounded in $L^\infty([0, T]\,; \mathbb{R}^n)$).

Since for every $\eta \in (0, \eta_i]$ one has

$$\forall n \in \mathbb{N},\ |\Lambda_n^2(\eta)| \leq \int_{\tau_i - \eta}^{\tau_i + \eta} h_n(\varphi(x_n(t)))|g(x^*(t)) - g(x^*(\tau_i))|\, \mathrm{d}t \leq \int_{\tau_i - \eta_i}^{\tau_i + \eta_i} h_n(\varphi(x_n(t)))|g(x^*(t)) - g(x^*(\tau_i))|\, \mathrm{d}t,$$

we obtain the following property:

$$\exists \eta_i > 0,\ \forall n \in \mathbb{N},\ \forall \eta \in (0, \eta_i],\ |\Lambda_n^2(\eta)| \leq \frac{\varepsilon}{3}. \tag{3.14}$$

Now, the sequence $(x_n)_n$ uniformly converges to x^* over $[0, T]$. Hence, there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$, one has $|g(x_n(t)) - g(x^*(t))| \leq \frac{\varepsilon}{3M}$ for every $t \in [0, T]$. This gives us the following property:

$$\exists N' \in \mathbb{N},\ \forall n \geq N',\ \forall \eta \in (0, \delta],\ |\Lambda_n^1(\eta)| \leq \frac{\varepsilon}{3}. \tag{3.15}$$

The last step is to apply Lemma 3.1 for $\eta = \eta_i$ which provides the inequality:

$$\exists N'' \in \mathbb{N},\ \forall n \geq N'',\ |(\tilde{I}_{i,n}(\eta_i) - \ell_i) g(x^*(\tau_i))| \leq \frac{\varepsilon}{3}. \tag{3.16}$$

Let us set $N := \max(N', N'')$. Combining (3.14)-(3.15)-(3.16) then gives (3.12). $\square$

[6] We refer to a classical property which asserts that given a sequence of mollifier $(f_n)_n$ defined over $[0, 1]$ and a continuous function $g : [0, 1] \to \mathbb{R}$, then $\int_0^1 f_n(t) g(t)\, \mathrm{d}t \to g(0)$ when n goes to infinity.

Let us underline that for an arbitrary sequence $(x_n)_n$ satisfying (2.2), which converges (up to a sub-sequence) to a solution $\bar{x}$ of (2.2), the sequence of integrals $(I_n)_n$ is not necessarily bounded even if the limiting trajectory $\bar{x}$ has a transverse crossing time (see example below).

Example 3.1. *Consider the scalar dynamics*

$$\dot{x}(t) = u(t) \quad \text{a.e. } t \in [0,2],$$

where $u(t) \in [0,1]$, together with the set $K := \mathbb{R}_-$ (associated with $\varphi(x) := x$). As nominal path, we consider $\bar{x}(t) := t-1$, $t \in [0,2]$. Observe that the function $\bar{\rho}(t) = \varphi(\bar{x}(t))$ is differentiable with $\bar{\rho}'(t) = 1 > 0$ for every $t \in [0,2]$, thus $\tau = 1$ is a transverse crossing time. For this example, we suppose for convenience that the mollifier $(h_n)_n$ is such that $a_n = 0$ and $b_n > 0$ for every $n \in \mathbb{N}$. Let us denote by $c_n \in (a_n, b_n)$ the unique point at which h_n achieves its maximum (one has $h_n(c_n) \to +\infty$ whenever $n \to +\infty$). Next, we consider the sequence of absolutely continuous function $(x_n)_n$ defined as

$$\begin{cases} x_n(t) = & t - 1 + \zeta_n, \quad t \in [0, 1-\xi_n], \\ x_n(t) = & c_n, \quad t \in [1-\xi_n, 1+\xi_n'], \\ x_n(t) = & t - 1 - \zeta_n', \quad t \in [1+\xi_n', 2], \end{cases} \tag{3.17}$$

where $(\zeta_n)_n$, $(\zeta_n')_n$ converge to 0 when $n \to +\infty$, $(\xi_n)_n$ and $(\xi_n')_n$ are with values in $(0,1)$, and:

$$\forall n \in \mathbb{N}, \quad \zeta_n - \xi_n = c_n, \quad \xi_n' - \zeta_n' = \sigma_n. \tag{3.18}$$

In addition, $(\sigma_n)_n$ is a sequence converging to 0 as $n \to +\infty$. Equality (3.18) guarantees the continuity of x_n at $t = 1-\xi_n$ and at $t = 1+\xi_n'$ for every $n \in \mathbb{N}$. Sequences $(\xi_n)_n$ and $(\xi_n')_n$ will be made more precise hereafter and, so, (3.18) allows to uniquely define the sequences $(\zeta_n)_n$ and $(\zeta_n')_n$.

Suppose now that $(\xi_n)_n$ and $(\xi_n')_n$ are chosen such that $(\xi_n + \xi_n')h_n(c_n) \to +\infty$ whenever $n \to +\infty$. Then, we we can easily see that $I_n \to +\infty$. Indeed, I_n can be written $I_n = I_n^1 + I_n^2 + I_n^3$ with

$$I_n^1 := \int_0^{1-\xi_n} h_n(\varphi(x_n(t)))\,\mathrm{d}t\ ;\ I_n^2 := \int_{1-\xi_n}^{1+\xi_n'} h_n(\varphi(x_n(t)))\,\mathrm{d}t\ ;\ I_n^3 := \int_{1+\xi_n'}^{2} h_n(\varphi(x_n(t)))\,\mathrm{d}t.$$

Recall that $\varphi(x) = x$, thus by changing the integration variable t into $s := t - 1 + \zeta_n$, resp. $s := t - 1 - \zeta_n'$ in I_n^1, resp. in I_n^3, we obtain that for every $n \in \mathbb{N}$, one has $I_n^1 \leq 1$ and $I_n^3 \leq 1$. Now, by construction, for every $n \in \mathbb{N}$, we have

$$I_n^2 = h_n(c_n)(\xi_n + \xi_n'),$$

which shows that $I_n \to +\infty$. In addition, we easily check that the sequence $(x_n)_n$ strongly-weakly converges to $\bar{x}$. First, one has

$$\sup_{t \in [0,T]} |x_n(t) - \bar{x}(t)| \leq \max(\zeta_n, \zeta_n', \sigma_n) \to 0,$$

whenever $n \to +\infty$. Second, one can verify that $(\dot{x}_n)$ converges a.e. to $\dot{\bar{x}}$ (actually for every $t \in [0,2]\backslash\{1\}$) which is enough to ensure the weak convergence of $(\bar{x}_n)_n$ to $\dot{\bar{x}}$ in $L^2([0,2]\,;\mathbb{R})$.

Nevertheless, we shall see later on that this phenomenon does not occur whenever $(x_n)_n$ is a minimizing sequence obtained from (2.5). This is due to the application of Pontryagin's Principle that provides additional properties on the extremal (x_n, p_n, u_n) preventing $(I_n)_n$ to blow up under (H′).

4 Optimality conditions for the time crisis problem

In this section, we give optimality conditions without the HMP, *i.e.*, by passing to the limit into the state-adjoint system satisfies by the extremal (x_n, p_n, u_n). Let us start by giving a definition of a covector associated with Problem (2.1). Doing so, we define the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ associated with (2.1) as

$$H(x,p,u) = p \cdot f(x,u) - \mathbb{1}_{K^c}(x).$$

Definition 4.1. *Given a solution (x^*, u^*) of (2.1) with r crossing times, we say that a piecewise absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ is a covector associated to x^* if p is absolutely continuous on $[0,T]\backslash\{\tau_1, ..., \tau_r\}$ and satisfies the following conditions.*

- *The function p fulfills the backward adjoint equation:*

$$\left| \begin{array}{ll} \dot{p}(t) & = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{a.e. } t \in [0,T], \\ p(T) & = 0. \end{array} \right. \tag{4.19}$$

- *The Hamiltonian condition is fulfilled almost everywhere:*

$$\forall u \in U,\ p(t) \cdot f(x^*(t), u) \leq p(t) \cdot f(x^*(t), u^*(t)) \quad \text{a.e. } t \in [0,T]. \tag{4.20}$$

- *At every crossing time, p admits left and right limits, i.e.,*

$$\forall i \in \{1, ..., r\},\ \exists p(\tau_i^\pm) := \lim_{t \to \tau_i^\pm} p(t). \tag{4.21}$$

- *There exists $(\ell_1, ..., \ell_r) \in (0, +\infty)^r$ such that:*

$$\forall i \in \{1, ..., r\}, \quad p(\tau_i^+) - p(\tau_i^-) = \ell_i \nabla\varphi(x^*(\tau_i)). \tag{4.22}$$

- *The Hamiltonian is constant almost everywhere over $[0,T]$, i.e., there is $\tilde{H} \in \mathbb{R}$ such that*

$$\tilde{H} = H(x(t), p(t), u(t)) = \max_{u \in U} H(x(t), p(t), u) = -\mathbb{1}_{K^c}(x^*(T)) \quad \text{a.e. } t \in [0,T]. \tag{4.23}$$

We call extremal of (2.1) *any triple (x^*, p, u^*) satisfying* (2.2)*,*(4.19)*-*(4.20)*,* (4.22) *and* (4.23)*.*

Remark 4.1. *Equality* (4.22) *amounts to say that p has a jump at $t = \tau_i$ in the normal cone to the set K (meaning here that the jump is in the direction of $\nabla\varphi(x(\tau_i))$, being assumed that K has a smooth boundary).*

To establish optimality conditions for (2.1), we start by proving the convergence of $(p_n)_n$. Doing so, we proceed step by step.

Lemma 4.1. *Suppose that $(p_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$ and let $1 \leq i \leq r$, $\eta \in (0, \delta]$. Then, up to a sub-sequence, the pair $(x_n, u_n)_n$ strongly-weakly converges over $[\tau_i + \eta, \tau_{i+1} - \eta]$ to a solution of*

$$\left| \begin{array}{ll} \dot{x}^*(t) & = f(x^*(t), u^*(t)) \\ \dot{p}(t) & = -D_x f(x^*(t), u^*(t))^\top p(t) \end{array} \right. \quad \text{a.e. } t \in [\tau_i + \eta, \tau_{i+1} - \eta].$$

Proof. In view of Property 3.1, for n large enough, the triple (x_n, u_n) satisfies the system

$$\left| \begin{array}{ll} \dot{x}_n(t) & = f(x_n(t), u_n(t)) \\ \dot{p}_n(t) & = -D_x f(x_n(t), u_n(t))^\top p_n(t) \end{array} \right. \quad \text{a.e. } t \in [\tau_i + \eta, \tau_{i+1} - \eta].$$

Let $t_0 \in [\tau_i + \eta, \tau_{i+1} - \eta]$. Since $(p_n)_n$ is bounded, we may assume that $(p_n(t_0))_n$ converges (up to a sub-sequence). Because $(x_n)_n$ uniformly converges to x^*, we deduce that $x_n(t_0) \to x^*(t_0)$. Using (H1)-(H2), the result of compactness of solutions of a control system (see, *e.g.*, [11, Theorem 1.11]) yields the result. $\square$

Next, we show that $(p_n)_n$ strongly-weakly converges on $[0,T] \setminus \{\tau_1, ..., \tau_r\}$.

Lemma 4.2. *Suppose that $(p_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. Then, there exists a function $p : [0,T] \to \mathbb{R}^n$ absolutely continuous on $[0,T] \backslash \{\tau_1, ..., \tau_r\}$ satisfying*

$$\dot{p}(t) = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{a.e. } t \in [0,T], \tag{4.24}$$

such that for every $\eta \in (0, \delta)$, $(p_n)_n$ strongly-weakly converges to p over $\mathcal{J}_\eta$.

Proof. Let $\eta \in (0, \delta)$. By using the previous lemma, for every $1 \leq i \leq r$, we obtain the existence of an absolutely continuous function p_η defined over $\mathcal{J}_\eta$ and satisfying (4.24) over $\mathcal{J}_\eta$. We now argue that p_η does not depend on η by considering a sequence of positive numbers $(\eta_k)_k$ such that $\eta_k \downarrow 0$ which allows us to define p_{η_k} for every $k \in \mathbb{N}$, as previously (over $\mathcal{J}_{\eta_k}$). We then obtain $p_{\eta_{k+1}}(t) = p_{\eta_k}(t)$ for every $t \in \mathcal{J}_{\eta_k}$ because $\mathcal{J}_{\eta_k} \subset \mathcal{J}_{\eta_{k+1}}$ for every $k \in \mathbb{N}$. This shows that we can then define a function $p : [0,T] \backslash \{\tau_1, ..., \tau_r\} \to \mathbb{R}^n$ without any ambiguity by the equality $p = p_\eta$ on every set $\mathcal{J}_\eta$, $\eta \in (0, \delta]$. By construction, p does not depend on η, which is as wanted. $\square$

Let us now address the question of the constancy of the Hamiltonian along (x^*, p, u^*) and the Hamiltonian condition (4.20).

Lemma 4.3. *Suppose that* $(p_n)_n$ *is uniformly bounded in* $L^\infty([0,T]\,;\mathbb{R}^n)$. *Then, the function* p *satisfies the Hamiltonian condition* (4.20) *and* (4.23).

Proof. Recall that x^* has r crossing times τ_i, $i=1,...,r$. Let $0 \le i \le r$ and $t_0 \in (\tau_i, \tau_{i+1})$ be a Lebesgue point of the measurable function $t \mapsto p(t) \cdot \dot{x}^*(t)$. From (2.7), we obtain for any $u \in U$ and $\nu > 0$ (small enough):

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot f(x_n(t),u)\,\mathrm{d}t \le \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot \dot{x}_n(t)\,\mathrm{d}t.$$

Now, $(p_n)_n$ and $(x_n)_n$ uniformly converge over $[t_0, t_0+\nu]$ to p and x^* respectively. In addition, $(\dot{x}_n)_n$ weakly converges to $\dot{x}^*$ over $[t_0, t_0+\nu]$. It follows that

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot f(x^*(t),u)\,\mathrm{d}t \le \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot \dot{x}^*(t)\,\mathrm{d}t.$$

Letting $\nu \downarrow 0$ then gives

$$p(t_0)\cdot f(x(t_0),u) \le p(t_0)\cdot \dot{x}^*(t_0).$$

Since $u \in U$ is arbitrary and almost every point of $[0,T]$ is a Lebesgue point of $t \mapsto p(t)\cdot \dot{x}^*(t)$, we obtain (4.20).

We proceed similarly to show the constancy of H over time, *i.e.*, for showing (4.23). Since the Hamiltonian H_n is autonomous for any $n \in \mathbb{N}$, one has

$$\tilde{H}_n := \max_{u\in U} H_n(x_n(t), p_n(t), u) = -G_n(\varphi(x_n(T)), \quad \forall t \in [0,T]$$

and as $G_n(\varphi(x_n(T)) \in [0,1]$, $(\tilde{H}_n)_n$ converges, up to a sub-sequence, to a constant $\tilde{H} \in [-1,0]$. Let $i \in \{0,...,r\}$ and again, let $t_0 \in (\tau_i, \tau_{i+1})$ be a Lebesgue point of $t \mapsto p(t)\cdot \dot{x}^*(t)$. For $\nu > 0$ small enough, one has:

$$\tilde{H}_n = \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot \dot{x}_n(t)\,\mathrm{d}t - \frac{1}{\nu}\int_{t_0}^{t_0+\nu} G_n(\varphi(x_n(t)))\,\mathrm{d}t.$$

By letting $n \to +\infty$, we deduce that

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} p_n(t)\cdot \dot{x}_n(t)\,\mathrm{d}t \to \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot \dot{x}^*(t)\,\mathrm{d}t.$$

By the uniform convergence of $(x_n)_n$ over $[t_0, t_0+\nu] \subset (\tau_i, \tau_{i+1})$, we also have

$$\frac{1}{\nu}\int_{t_0}^{t_0+\nu} G_n(\varphi(x_n(t)))\,\mathrm{d}t \to \frac{1}{\nu}\int_{t_0}^{t_0+\nu} \mathbb{1}_{K^c}(x^*(t))\,\mathrm{d}t,$$

when $n \to +\infty$ (this follows using the dominated convergence Theorem). We can then conclude that

$$\tilde{H} = \frac{1}{\nu}\int_{t_0}^{t_0+\nu} p(t)\cdot \dot{x}^*(t)\,\mathrm{d}t - \frac{1}{\nu}\int_{t_0}^{t_0+\nu} \mathbb{1}_{K^c}(x^*(t))\,\mathrm{d}t.$$

Now, letting $\nu \downarrow 0$ (recall that $t_0 \in (\tau_i, \tau_{i+1})$ for $i=1,...,r$) gives

$$\tilde{H} = p(t_0)\cdot \dot{x}(t_0) - \mathbb{1}_{K^c}(x(t_0)).$$

Since almost every point $t_0 \in [0,T]$ is a Lebesgue point of the map $t \mapsto p(t)\cdot \dot{x}^*(t)$, one has then

$$H(x^*(t), p(t), u^*(t)) = \tilde{H} \quad \text{a.e. } t \in [0,T].$$

By the Hamiltonian condition (4.20) and the continuity of the map $t \mapsto \max_{u\in U} H(x^*(t), p(t), u)$ on $[0,T] \setminus \{\tau_1, ..., \tau_r\}$, (4.23) is thus fullfilled (recall that $p(T) = 0$ and thus $\tilde{H} = -\mathbb{1}_{K^c}(x(T))$).

□

Thanks to the previous lemma, we can now give the main result of this section, namely that (x^*, p, u^*) is an extremal of Problem (2.1).

Theorem 4.1. *Suppose that the sequence of integrals $(I_n)_n$ is bounded. Then, there exists a non-null covector $p : [0,T] \to \mathbb{R}^n$ associated to x^* in the sense of Definition 4.1.*

Proof. Accordingly to Proposition 3.1, the sequence $(p_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. The existence of a function $p : [0,T]\backslash\{\tau_1, ..., \tau_r\} \to \mathbb{R}^n$ satisfying (4.19)-(4.20)-(4.23) follows from the previous lemma. Note that the condition $p(T) = 0$ because $p_n(T) = 0$ for all $n \in \mathbb{N}$ and $p(T) = \lim_{n\to+\infty} p_n(T)$ (note also that $x^*(T) \notin \partial K$ since $\tau_r < T$ is the last crossing time).

Let us now show (4.21). Since $(p_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$, there is $R \geq 0$ such that for every $n \in \mathbb{N}$ one has $\|p_n\|_{L^\infty([0,T]\,;\mathbb{R}^n)} \leq R$. Since for every $t \in [0,T]\backslash\{\tau_1, ..., \tau_r\}$, one has $p(t) = \lim_{n\to+\infty} p_n(t)$, we deduce that $\|p\|_{L^\infty([0,T]\,;\mathbb{R}^n)} \leq R$. Now, fix $1 \leq i \leq r$ and observe that

$$\dot{p}(t) = -D_x f(x^*(t), u^*(t))^\top p(t) \quad \text{a.e. } t \in (\tau_i, \tau_{i+1}).$$

Given $t_1, t_2 \in (\tau_i, \tau_{i+1})$, we can thus write:

$$|p(t_2) - p(t_1)| = \left| \int_{t_1}^{t_2} -D_x f(x^*(t), u^*(t))^\top p(t)\, \mathrm{d}t \right| \leq A|t_1 - t_2|,$$

where $A := R \sup_{t\in[0,T]} |D_x f(x^*(t), u^*(t))^\top p(t)|$. This inequality shows that $p(\cdot)$ satisfies the Cauchy criterion at $t = \tau_i^+$ which proves that the right limit $p(\tau_i^+)$ exists. Similarly, one obtains the existence of a left limit $p(\tau_i^-)$. We can repeat this argumentation at every crossing time τ_i which gives (4.21).

Let us now prove the jump formula (4.22). Doing so, consider a sequence $(\varepsilon_k)_k$ such that $\varepsilon_k \downarrow 0$ and let us apply Proposition 3.2 with $\nabla\varphi$ in place of g. For every $k \in \mathbb{N}$, there exist $\eta_k \in (0, \delta]$ and $N_k \in \mathbb{N}$ such that for every $n \geq N_k$, one has

$$\left| \int_{\tau_i-\eta_k}^{\tau_i+\eta_k} h_n(\varphi(x_n(t)))\nabla\varphi(x_n(t))\, \mathrm{d}t - \ell_i \nabla\varphi(x^*(\tau_i)) \right| \leq \varepsilon_k.$$

Notice that from Proposition 3.2, one has $\eta_k \to 0$ when $k \to +\infty$. Integrating (2.6) over $[\tau_i - \eta_k, \tau_i + \eta_k]$ yields

$$\forall n \in \mathbb{N},\ p_n(\tau_i + \eta_k) - p_n(\tau_i - \eta_k) = \int_{\tau_i-\eta_k}^{\tau_i+\eta_k} -D_x f(x_n(t), u_n(t))^\top p_n(t)\, \mathrm{d}t + \int_{\tau_i-\eta_k}^{\tau_i+\eta_k} h_n(\varphi(x_n(t)))\nabla\varphi(x_n(t))\, \mathrm{d}t.$$

Now, $t \mapsto D_x f(x_n(t), u_n(t)) p_n(t)$ is uniformly bounded over $[0,T]$ (say by a constant $B \geq 0$). It follows that

$$\forall n \geq N_k,\ |p_n(\tau_i + \eta_k) - p_n(\tau_i - \eta_k) - \ell_i \nabla\varphi(x^*(\tau_i))| \leq 2B\eta_k + \varepsilon_k.$$

First, we let n goes to infinity (k being fixed) which gives:

$$\forall k \in \mathbb{N}, \quad |p(\tau_i + \eta_k) - p(\tau_i - \eta_k) - \ell_i \nabla\varphi(x^*(\tau_i))| \leq 2B\eta_k + \varepsilon_k.$$

Now, we let $k \to +\infty$ observing that $p(\tau_i \pm \eta_k) \to p(\tau_i^\pm)$ and we obtain

$$p(\tau_i^+) - p(\tau_i^-) = \ell_i \nabla\varphi(x^*(\tau_i)),$$

which is the desired property.

The last step is to show that for every $1 \leq i \leq r$, one has $\ell_i \neq 0$. Consider the map

$$h(t) := \max_{u\in U} H(x^*(t), p(t), u), \quad t \in [0,T] \setminus \{\tau_1, ..., \tau_r\}$$

which is continuous on each time interval (τ_{i-1}, τ_i). As p admits left and right limits at each τ_i, so is h. Consider $i \in \{1, ..., r\}$ such that x^* crosses ∂K from K to K^c at $t = \tau_i$. One has then

$$h(\tau_i^-) = \max_{u\in U} p(\tau_i^-) \cdot f(x^*(t), u), \quad h(\tau_i^+) = \max_{u\in U} p(\tau_i^+) \cdot f(x^*(t), u) - 1.$$

If $\ell_i = 0$, one has $p(\tau_i^+) = p(\tau_i^-)$ and thus $h(\tau_i^+) - h(\tau_i^-) = -1$, which contradicts property (4.23). Similarly, if x^* crosses ∂K from K^c to K at τ_i with $\ell_i = 0$, one gets $h(\tau_i^+) - h(\tau_i^-) = 1$ and again a contradiction with (4.23). □

Let us stress that this result does not involve the transverse assumption (H'). Using the constancy of the Hamiltonian along an extremal, the jump formula can be also written as follows (see also [17]).

Corollary 4.1. *Assume that the sequence of integrals $(I_n)_n$ is bounded. If $\dot{x}^*$ admits left and right derivative at some transverse crossing time τ_i, $i \in \{1, ..., r\}$, then the jump of the covector p at τ_i can be written as:*

$$p(\tau_i^+) = p(\tau_i^-) + \frac{\delta_i + p(\tau_i^-) \cdot (\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+)} \nabla\varphi(x^*(\tau_i)),$$

or

$$p(\tau_i^-) = p(\tau_i^+) - \frac{\delta_i + p(\tau_i^+) \cdot (\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-)} \nabla\varphi(x^*(\tau_i)),$$

where $\delta_i = +1$ resp. $\delta_i = -1$ if the crossing time τ_i is outward, resp. inward.

Proof. Let us write the conservation of the Hamiltonian in a right and left neighborhood of τ_i. If τ_i is an outward crossing time, one has then:

$$p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) - 1 = p(\tau_i^+)\dot{x}^*(\tau_i^+), \tag{4.25}$$

and $p(\tau_i^+) = p(\tau_i^-) + \ell_i \nabla\varphi(x^*(\tau_i))$. Replacing $p(\tau_i^+)$ by this last expression in equation (4.25) raises

$$\ell_i = \frac{1 + p(\tau_i^-).(\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^+))}{\nabla\varphi(x^*(\tau_i)).\dot{x}^*(\tau_i^+)}.$$

Similarly, replacing $p(\tau_i^-)$ by $p(\tau_i^+) - \ell_i \nabla\varphi(x^*(\tau_i))$ in equation (4.25) gives the equivalent expression

$$\ell_i = \frac{1 + p(\tau_i^+).(\dot{x}^*(\tau_i^-) - \dot{x}^*(\tau_i^-))}{\nabla\varphi(x^*(\tau_i)).\dot{x}^*(\tau_i^+)}.$$

If τ_i is an inward crossing time, one can easily check that this amounts to replace 1 by -1 in (4.25). $\square$

5 Sufficient conditions for the boundedness of the sequence $(I_n)_n$

The aim of this section is to give sufficient conditions on the system that ensure the boundedness of $(I_n)_n$. In that case, optimality conditions for an optimal path are guaranteed by Theorem 4.1. Given an optimal solution (x^*, u^*) of Problem (2.1), let us introduce the following hypothesis (in the spirit of the Hybrid Maximum Principle that requires also a transverse assumption [17]).

(H′) Every crossing time of x^* is transverse.

This hypothesis excludes the cases where the optimal solution x^* hits the boundary of K tangentially, *i.e.*,

$$\lim_{t \to \tau^+} \nabla\varphi(x(t)) \cdot \dot{x}(t) = 0 \quad \text{or} \quad \lim_{t \to \tau^-} \nabla\varphi(x(t)) \cdot \dot{x}(t) = 0, \tag{5.26}$$

at a crossing time τ. Actually, several cases could appear depending if both scalar products are zero in (5.26) or only one. As far as we know, the obtention of necessary conditions in this case has been few considered in the literature (except in [4]) and remains a thorough open question.

5.1 The transverse case

We start by the following result which covers the case where every crossing times of x^* are transverse.

Proposition 5.1. *Under Hypothesis* (H′)*, the sequence $(I_n)_n$ is bounded.*

Proof. From Proposition 3.1, we only need to show that $(I_n)_n$ is bounded. Suppose by contradiction that this is not the case. Extracting a sequence if necessary, we may assume that $I_n \to +\infty$ whenever $n \to +\infty$. Observe that the function $q_n := \frac{p_n}{I_n}$ satisfies the differential system

$$\left| \begin{array}{ll} \dot{q_n}(t) & = -D_x f(x_n(t), u_n(t))^\top q_n(t) + \tilde{h}_n(\varphi(x_n(t)))\nabla\varphi(x_n(t)) \quad \text{a.e. } t \in [0, T], \\ q_n(T) & = 0, \end{array} \right.$$

where $\tilde{h}_n(\sigma) := \frac{h_n(\sigma)}{I_n}$, $\sigma \in \mathbb{R}$. By this change of variable, one has obviously

$$\forall n \in \mathbb{N}, \int_0^T \tilde{h}_n(\varphi(x_n(t)))\, \mathrm{d}t = 1, \tag{5.27}$$

so, Proposition 3.1 implies that the sequence $(q_n)_n$ is uniformly bounded in $L^\infty([0,T]\,;\mathbb{R}^n)$. One can then repeat the same argumentation than in the proof of Theorem 4.1 on the sequence $(q_n)_n$, excepted the last point about the value of the Hamiltonian. Indeed, as the covector p_n has been renormalized, we have no longer the value of the Hamiltonian equal to $-G_n(\varphi(x_n(T))$. However, we obtain that there exists a piecewise absolutely continuous function $q:[0,T]\backslash\{\tau_1,...,\tau_r\}\to\mathbb{R}^n$ satisfying the following properties:

- Up to a sub-sequence, $(q_n)_n$ converges to q on every compact set of $[0,T]$ excluding $\{\tau_1,...,\tau_r\}$.
- The function q satisfies

$$\left|\begin{array}{ll}\dot{q}(t) & = -D_x f(x^*(t),u^*(t))^\top q(t) \quad \text{a.e. } t\in[0,T],\\ q(T) & =0.\end{array}\right.$$

- The Hamiltonian condition is fulfilled almost everywhere:

$$\forall u\in U,\ q(t)\cdot f(x^*(t),u)\leq q(t)\cdot f(x^*(t),u^*(t)) \quad \text{a.e. } t\in[0,T]. \tag{5.28}$$

- For every $1\leq i\leq r$, q admits a limit at $\tau_i^\pm$, *i.e.*, there exists $\lim_{t\to\tau_i^\pm} q(t)$.
- At every crossing time τ_i, $1\leq i\leq r$, there exist a scalar $\tilde{\ell}_i$ such that

$$q(\tau_i^+)-q(\tau_i^-)=\tilde{\ell}_i\nabla\varphi(x^*(\tau_i)). \tag{5.29}$$

Here, we can no longer guarantee that each scalar $\tilde{\ell}_i$ is non null. However, one has from Proposition 3.2, for every $1\leq i\leq r$ and every $\eta\in(0,\delta]$

$$\tilde{\ell}_i=\lim_{n\to+\infty}\int_{\tau_i-\eta}^{\tau_i+\eta}\tilde{h}_n(\varphi(x_n(t)))\,\mathrm{d}t. \tag{5.30}$$

Observe that

$$\forall n\in\mathbb{N},\ \frac{\tilde{H}_n}{I_n}=q_n(t)\cdot f(x_n(t),u_n(t))-\frac{G_n(\varphi(x_n(t)))}{I_n} \quad \text{a.e. } t\in[0,T].$$

For every $t\in[0,T]$, one has $\frac{\tilde{H}_n}{I_n}\to 0$ and $-\frac{G_n(\varphi(x_n(t)))}{I_n}\to 0$ when $n\to\infty$ because $\tilde{H}_n\in[-1,0]$ and $G_n(\varphi(x_n(t)))\in[0,1]$ for every $n\in\mathbb{N}$ and every $t\in[0,T]$. We then obtain that the covector q satisfies:

$$q(t)\cdot f(x^*(t),u^*(t))=0 \quad \text{a.e. } t\in[0,T], \tag{5.31}$$

by considering Lebesgue points of the map $t\mapsto q(t)\cdot f(x^*(t),u^*(t))$ and repeating exactly the same argumentation as in the proof of Lemma 4.3.

We claim now that $q\not\equiv 0$, *i.e.*, $q(\cdot)$ is non-null over $[0,T]$. To show this property, it is enough to prove that there exists $1\leq i\leq r$ such that $\tilde{\ell}_i\neq 0$. Suppose then by contradiction that for every $1\leq i\leq r$, one has $\tilde{\ell}_i=0$. By using Lemma 3.1 with $\tilde{h}_n$ in place of h_n, we obtain that

$$\forall i\in\{1,...,r\},\forall\eta\in(0,\delta],\ \lim_{n\to+\infty}\int_{\tau_i-\eta}^{\tau_i+\eta}\tilde{h}_n(\varphi(x_n(t)))\,\mathrm{d}t=0, \tag{5.32}$$

where $\eta\in(0,\delta]$ is fixed. From Property 3.1 one also has

$$\lim_{n\to+\infty}\int_{J_\eta}h_n(\varphi(x_n(t)))\,\mathrm{d}t=\lim_{n\to+\infty}\int_{J_\eta}\tilde{h}_n(\varphi(x_n(t)))\,\mathrm{d}t=0. \tag{5.33}$$

Combining (5.32) and (5.33) gives us a contradiction with (5.27), thus we have obtained that there is $1\leq i\leq r$ such that $\tilde{\ell}_i\neq 0$.

To conclude the proof of the proposition, we will finally exhibit a contradiction involving optimality conditions (5.29) and (5.31) satisfied by the covector q. Fix $1\leq i\leq r$ such that $\tilde{\ell}_i\neq 0$. First, using (5.28)-(5.31) at $t=\tau_i^-$, one has

$$q(\tau_i^-)\cdot\dot{x}^*(\tau_i^+)\leq q(\tau_i^-)\cdot\dot{x}^*(\tau_i^-)=0.$$

Combining with (5.29) gives

$$q(\tau_i^+)\cdot\dot{x}^*(\tau_i^+)-q(\tau_i^-)\cdot\dot{x}^*(\tau_i^+)=\tilde{\ell}_i\nabla\varphi(x^*(\tau_i))\cdot\dot{x}^*(\tau_i^+).$$

It follows that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) = -q(\tau_i^-) \cdot \dot{x}^*(\tau_i^+) \geq 0$$

Because $\tilde{\ell}_i \neq 0$ and $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) \neq 0$, we deduce that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) > 0. \tag{5.34}$$

We now proceed with the same reasoning using (5.28)-(5.31) at $t = \tau_i^+$. We obtain

$$q(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) \leq q(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = 0.$$

Combining with (5.29) gives

$$q(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) - q(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = \tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-).$$

It follows that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) = q(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) \leq 0.$$

Using again that $\tilde{\ell}_i \neq 0$ and $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) \neq 0$, we deduce that

$$\tilde{\ell}_i \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) < 0. \tag{5.35}$$

We claim that (5.34)-(5.35) is a contradiction. Indeed, because $\tilde{\ell}_i$ is non-zero, (5.34)-(5.35) imply that the scalar products $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-)$ and $\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+)$ are of opposite sign. Because at $t = \tau_i$, the trajectory crosses the boundary of K, we necessarily have that

$$\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) > 0 \quad \text{and} \quad \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) > 0,$$

if τ_i is an outward crossing time, or

$$\nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^-) < 0 \quad \text{and} \quad \nabla\varphi(x^*(\tau_i)) \cdot \dot{x}^*(\tau_i^+) < 0,$$

if it is inward. This contradiction completes the proof of the proposition and shows that $(I_n)_n$ is necessarily a bounded sequence. □

5.2 A weaker condition

We give now conditions on the subsequence $(x_n)_n$, and not on the limiting solution x^*, that ensures the boundedness of the sequence of integrals $(I_n)_n$. For $n \in \mathbb{N}$, define an absolutely continuous function ρ_n as:

$$\rho_n(t) := \varphi(x_n(t)) \quad t \in [0, T].$$

We known that for each $n \in \mathbb{N}$, x_n is almost everywhere differentiable on $[0, T]$ an thus ρ_n as well with

$$\dot{\rho}_n(t) = \nabla\varphi(x_n(t))\dot{x}_n(t) \quad \text{a.e. } t \in [0, T].$$

In addition, $(\rho_n)_n$ is uniformly bounded in $L^\infty([0, T]\,; \mathbb{R})$ thanks to (H1) and (H3). Therefore, for $i \in \{1, ..., r\}$ and $n \in \mathbb{N}$, we can define:

$$l_{i,n}^+ := \limsup_{h\to 0} \operatorname*{ess\,sup}_{t\in[\tau_i - h, \tau_i + h]} \dot{\rho}_n(t) \quad ; \quad l_{i,n}^- := \liminf_{h\to 0} \operatorname*{ess\,inf}_{t\in[\tau_i - h, \tau_i + h]} \dot{\rho}_n(t).$$

Remark 5.1. *In many optimal control problems, optimal solutions x_n are piecewise C^1, and thus the function ρ_n admits left and right derivatives at any $t \in (0, T)$. Then, the definition of the numbers $l_{i,n}^\pm$ simply involves the maximum and minimum of $\dot{\rho}_n(\tau_i^\pm)$.*

Proposition 5.2. *If for every $1 \leq i \leq r$, one has*

$$\begin{aligned} &\liminf_{n\to+\infty} l_{i,n}^- > 0 \text{ if } \tau_i \text{ is an outward crossing time, or} \\ &\limsup_{n\to+\infty} l_{i,n}^+ < 0 \text{ if } \tau_i \text{ is an inward crossing time,} \end{aligned} \tag{5.36}$$

then the sequence $(I_n)_n$ is bounded.

Proof. For $i = 1, ..., r$, set

$$\underline{l}_i = \liminf_{n\to+\infty} l_{i,n}^-, \quad \bar{l}_i = \limsup_{n\to+\infty} l_{i,n}^+,$$

and define the numbers η_i^-, η_i^+ $(i = 1, ..., r)$ as

$$\eta_i^+ = \eta_{i+1}^- := \frac{1}{2}(\tau_{i+1} - \tau_i) \quad 1 \le i \le r-1,$$

together with $\eta_1^- := \tau_1$ and $\eta_r^+ := T - \tau_r$. As well, let us define the integrals:

$$\tilde{I}_{i,n}^-(\eta) := \int_{\tau_i-\eta}^{\tau_i} h_n(\varphi(x_n(t))\,\mathrm{d}t, \quad \tilde{I}_{i,n}^+(\eta) := \int_{\tau_i}^{\tau_i+\eta} h_n(\varphi(x_n(t))\,\mathrm{d}t \quad i = 1, ..., r.$$

One has clearly

$$I_n = \sum_{i=1}^{r} \left[\tilde{I}_{i,n}^-(\eta_i^-) + \tilde{I}_{i,n}^+(\eta_i^+)\right].$$

We show now that for any $i \in \{1, ..., r\}$ such that τ_i is an outward crossing time, the sequence $(\tilde{I}_{i,n}^-(\eta_i^-))_n$ is bounded. Observe first that one has:

$$\begin{aligned}
\underline{l}_i \tilde{I}_{i,n}^-(\eta_i^-) &= \int_{\tau_i-\eta_i^-}^{\tau_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t + \int_{\tau_i-\eta_i^-}^{\tau_i} h_n(\rho_n(t))\dot{\rho}_n(t)\,\mathrm{d}t \\
&= \int_{\tau_i-\eta_i^-}^{\tau_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t + G_n(\varphi(x_n(\tau_i))) - G_n(\varphi(x_n(\tau_i - \eta_i^-))).
\end{aligned} \tag{5.37}$$

Let $\varepsilon \in (0, \underline{l}_i)$. We claim that there exist $\zeta_i \in (0, \eta_i^-)$ and $N > 0$ such that

$$\forall n \ge N,\ \underline{l}_i < \dot{\rho}_n(t) + \varepsilon \quad \text{a.e. } t \in [\tau_i - \zeta_i, \tau_i]. \tag{5.38}$$

Indeed, by definition of $\underline{l}_i$, there exists $N \in \mathbb{N}$ such that for every $n \ge N$ one has

$$\underline{l}_i \le l_{i,n}^- + \frac{\varepsilon}{2}.$$

We also have for every $h > 0$ sufficiently small:

$$\forall n \in \mathbb{N}, \operatorname*{ess\,inf}_{s\in[\tau_i-h,\tau_i+h]} \dot{\rho}_n(s) \le \dot{\rho}_n(t) \quad \text{a.e. } t \in [\tau_i - h, \tau_i + h].$$

By definition of $l_{i,n}^-$, we deduce that there exists $\zeta_i \in (0, \eta_i^-)$ such that

$$l_{i,n}^- \le \operatorname*{ess\,inf}_{s\in[\tau_i-\zeta_i,\tau_i+\zeta_i]} \dot{\rho}_n(s) + \frac{\varepsilon}{2},$$

and thus, we obtain (5.38). One can then write

$$\begin{aligned}
\int_{\tau_i-\eta_i^-}^{\tau_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t &< \varepsilon \int_{\tau_i-\zeta_i}^{\tau_i} h_n(\rho_n(t))\,\mathrm{d}t + \int_{\tau_i-\eta_i^-}^{\tau_i-\zeta_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t \\
&< \varepsilon \tilde{I}_{i,n}^-(\eta_i^-) + \int_{\tau_i-\eta_i^-}^{\tau_i-\zeta_i} h_n(\rho_n(t))(\underline{l}_i - \dot{\rho}_n(t))\,\mathrm{d}t.
\end{aligned} \tag{5.39}$$

Observe now that the scalar m defined as

$$m := \min_{t\in[\tau_i-\eta_i^-,\tau_i-\zeta_i]} \varphi(x^*(t)),$$

is such that $m < 0$ since $x^*(t)$ belongs to K for $t \in [\tau_i - \eta_i^-, \tau_i - \zeta_i]$. From the uniform convergence of the sequence $(x_n)_n$ to x^* over $[0, T]$ and since φ is continuous, there exists $N' \ge N$ such that one has $\rho_n(t) < -m/2$ for any $n > N'$ and every $t \in [\tau_i - \eta_i^-, \tau_i - \zeta_i]$. Then, from the convergence of the sequence of negative numbers $(a_n)_n$ to 0, there exists $\bar{N} \ge N'$ such that one has $a_n > -m/2$ for any $n > \bar{N}$. This implies that

$$\forall n > \bar{N},\ \forall t \in [\tau_i - \eta_i^-, \tau_i - \zeta_i],\ h_n(\rho_n(t)) = 0. \tag{5.40}$$

Finally, from (5.37), (5.39) and (5.40), one obtains

$$\forall n > \bar{N},\ \underline{l}_i \tilde{I}^-_{i,n}(\eta_i^-) < \varepsilon \tilde{I}^-_{i,n}(\eta_i^-) + 1.$$

This shows that the sequence $(\tilde{I}^-_{i,n}(\eta_i^-))_n$ is bounded. For $i \in \{1, ..., r\}$ such that τ_i is an inward crossing time, we proceed in the same way to show that the sequence $(\tilde{I}^-_{i,n}(\eta_i^-))_n$ is bounded, considering the number $\bar{l}_i < 0$. The proof of boundedness of the sequences $(\tilde{I}^+_{i,n}(\eta_i^+))_n$ is analogous. □

From a practical point of view, x^* and its crossing times τ_i are known only approximately. Whenever ρ_n admits left and right derivatives, condition (5.36) is merely guaranteed whenever $(\dot{\rho}_n(t^{\pm}))_n$ is bounded below by a positive number, or bounded above by a negative number, locally at each crossing time τ_i, $1 \leq i \leq r$.

5.3 A reciprocal property

We have seen previously that under (H'), the sequence $(I_n)_n$ is bounded (as well as under condition (5.36) which is weaker than (H')). Thanks to these conditions, we obtained optimality conditions for an optimal solution x^* (under the assumption that it has a finite number of crossing times). We now would like to address the converse question, namely, what can be said about x^* whenever the sequence $(I_n)_n$ is bounded? In that case, we can prove the following result.

Proposition 5.3. *Suppose that the sequence $(I_n)_n$ is bounded and let τ_i be a crossing time of x^* such that $\dot{x}^*(\tau_i^{\pm})$ exist. Then, if τ_i is an outward, resp. inward, crossing time, one has $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x(t_i)) \neq 0$, resp. $\dot{x}^*(t_i^-) \cdot \nabla\varphi(x(t_i)) \neq 0$.*

Proof. Since $(I_n)_n$ is bounded, there exists a (non-null) covector p as in Definition 4.1 (see Theorem 4.1). Consider now an outward crossing time τ_i. The conditions satisfied by the extremal (x^*, p, u^*) imply the jump of p at $t = \tau_i$ and the constancy of the Hamiltonian as in Definition 4.1. These conditions can be written

$$\left| \begin{array}{l} p(\tau_i^+) = p(\tau_i^-) + \ell_i \nabla\varphi(x^*(t_i)), \\ p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) - 1. \end{array} \right.$$

The Hamiltonian condition at $t = \tau_i$ also gives us the following inequalities:

$$\left| \begin{array}{l} p(\tau_i^-) \cdot f(x^*(\tau_i), u) \leq p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-), \\ p(\tau_i^+) \cdot f(x^*(\tau_i), u) \leq p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+), \end{array} \right.$$

for every $u \in U$. Suppose by contradiction that $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) = 0$. It follows that:

$$p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = p(\tau_i^-) \cdot \dot{x}^*(\tau_i^+)$$

Using the Hamiltonian condition, we obtain

$$p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = p(\tau_i^-) \cdot \dot{x}^*(\tau_i^+) \leq p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) - 1,$$

which is a contradiction. It follows that $\dot{x}^*(t_i^+) \cdot \nabla\varphi(x^*(t_i)) \neq 0$ as was to be proved.

In the case where τ_i is an inward crossing time, we proceed in the same way supposing by contradiction that $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) = 0$. In that case, the constancy of the Hamiltonian gives us

$$p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) - 1 = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+).$$

By a similar computation, we obtain using that $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) = 0$:

$$p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) = p(\tau_i^+) \cdot \dot{x}^*(\tau_i^-) \leq p(\tau_i^+) \cdot \dot{x}^*(\tau_i^+) = p(\tau_i^-) \cdot \dot{x}^*(\tau_i^-) - 1.$$

This is a contradiction, which ends the proof. □

This proposition shows that at every crossing time τ_i of x^* for which $\dot{x}(\tau_i^{\pm})$ exists, the trajectory is always transverse "at the exterior" of K, *i.e.*,

Case 1 : if τ_i is an outward crossing time, then $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) > 0$;

Case 2 : if τ_i is an inward crossing time, then $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) < 0$;

In both cases, we can only say that $\dot{x}^*(\tau_i^-) \cdot \nabla\varphi(x^*(\tau_i)) \geq 0$ (in case 1) or $\dot{x}^*(\tau_i^+) \cdot \nabla\varphi(x^*(\tau_i)) \leq 0$ (in case 2).

6 Numerical examples

To illustrate convergence of the sequence of ovectors $(p_n)_n$ to the covector p, we present next two examples based on the one to be found in [5], namely, we consider the system

$$\begin{cases} \dot{x}_1(t) &= -x_2(t)(2+u(t)), \\ \dot{x}_2(t) &= x_1(t)(2+u(t)), \end{cases} \tag{6.41}$$

with initial condition $(x_1(0), x_2(0)) = (0,1)$ and $u(t) \in [-1,1]$. In addition, define $\varphi_1, \varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ as

$$\varphi_1(x_1,x_2) := \frac{1}{2}x_1^2 + 2x_2^2 - 1 \quad ; \quad \varphi_2(x_1,x_2) := \varphi(x_1,x_2) = x_1^2 + 4\min(0,x_2)^2 - 1.$$

Next, we consider the two following situations. In the first one, optimal trajectories will hit the boundary of the set

$$K_1 := \{(x_1,x_2) \in \mathbb{R}^2 \ ; \ \varphi_1(x_1,x_2) \leq 0\},$$

transversely when entering and leaving this set. In the second one, optimal trajectories will hit the boundary of the set

$$K_2 := \{(x_1,x_2) \in \mathbb{R}^2 \ ; \ \varphi_2(x_1,x_2) \leq 0\},$$

tangentially. For the regularization scheme, we use a penalty method following arising from a reformulation of the time of crisis problem ([9]). This amounts then to consider the optimal control problem

$$\inf_{(u,v)\in\mathcal{U}\times\mathcal{V}} \int_0^T \left(\frac{1+v(t)}{2} + n\left[v(t)\varphi_i(x_1(t),x_2(t)) - |\varphi_i(x_1(t),x_2(t))|\right] \right) \mathrm{d}t, \tag{6.42}$$

where $i = 1,2$, $\mathcal{V} := \{v : [0,T] \to \{\pm 1\} \ ; \ v \text{ meas.}\}$, and $n \to +\infty$. We refer to [9] for more details about this method. In particular, it is proved that this penalization approach is equivalent to a regularization of the time of crisis problem with a particular sequence for $(G_n)_n$. We next wish to examine the behavior of covectors for various values of the penalized parameter n. Doins so, We use the numerical solver `BocopHJB` [18] to obtain informations about an optimal solution, and we then integrate the Hamiltonian system backward in time to compute covectors. Numerical results are depicted in Fig. 4 and Fig. 3 on which we distinguish the contrast between the two cases:

- In the transverse case, we observe a "good" convergence of covectors and also the boundedness of the sequence $(I_n)_n$.
- In the tangent case, we observe that the sequence of covectors becomes arbitrarily large when the parameter n increases. Similarly, we also observe that the sequence $(I_n)_n$ is taken large values when n increases.

These observations highlight the differences between the transverse case and the non-transverse case.

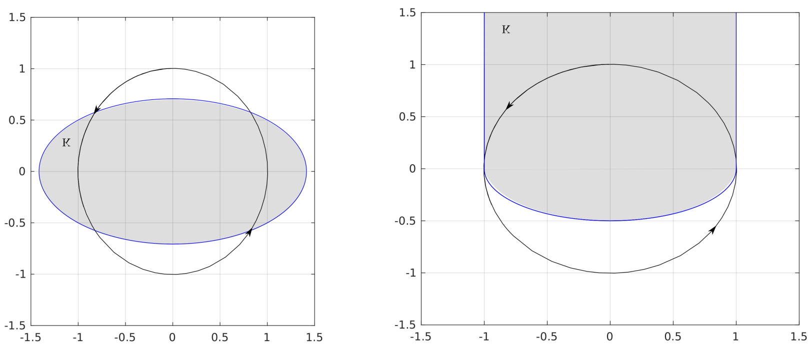

Figure 2: Optimal trajectories $x = (x_1, x_2)$ entering and leaving the set K_1 (fig. left) and the set K_2 (fig. right).

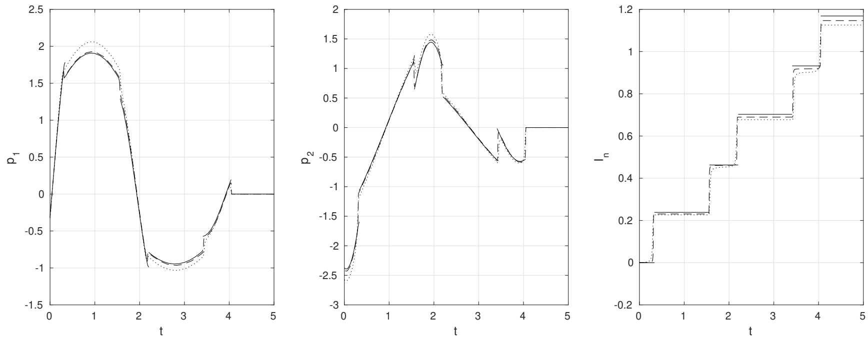

Figure 3: Convergence of covectors when crossing times are transverse (dotted: $n = 10$, dashed: $n = 10^2$, solid: hybrid covector). On fig. right, behavior of $(I_n)_n$ (bounded in this case).

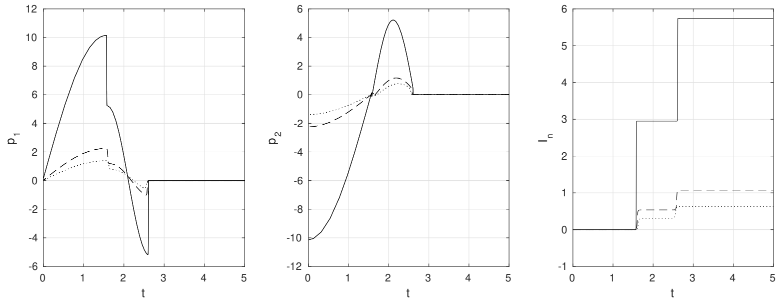

Figure 4: Behavior of covectors when crossing times are non-transverse (dotted: $n = 10$, dashed: $n = 10^2$, solid: $n = 10^5$). On fig. right, behavior of $(I_n)_n$.

7 Conclusion

Our proof of boundedness of the sequence of covectors presents some analogy with materials of the work [17]. However, we have treated the question of boundedness of the sequence of covectors via a suitable sequence of integrals $(I_n)_n$, and we also did not use at all the Hybrid Maximum Principle in this paper (which is a byproduct of this approach). As well, we also did not need to introduce needle-like perturbations and the variation vector to derive necessary optimality conditions. Note that we also obtain these conditions for the time crisis problem under a weaker condition than requiring transverse crossing times for the optimal trajectory.

Our methodology also presents several interest from a numerical point of view since we introduced a regularized optimal control problem in place of a discontinuous problem (which is more delicate to handle numerically). In addition, we introduced an auxiliary condition to guarantee necessary conditions for an optimal solution x^*. This condition involves an approximated sequence which can be tested numerically and not a solution x^* of the problem (that is unknown a priori).

Some interesting issues that are out of the scope of this paper could be the matter of future works. In particular, one would like to know more into details the link between the boundedness of the sequence $(I_n)_n$ and the behavior of x^* at a crossing time τ (*i.e.*, if x^* is transverse to the boundary of K at $t = \tau$ or not). The methodology that has been developed in this paper could also be used to obtain an extension of optimality conditions in the hybrid setting whenever an optimal path has a so-called *semi-transverse* crossing time, *i.e.*, whenever one and only one of the two scalar products in (5.26) is zero. As well, we believe that if x^* is purely transverse at some isolated crossing time (*i.e.*, both scalar products are zero in (5.26)), then $(I_n)_n$ is unbounded (but we have no proof of this result).

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