An upper bound for the number of chess diagrams without promotion

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Abstract

The number of legal chess diagrams without promotion is bounded from above by 2×10^{40}. This number is obtained by restricting both bishops and pawns position and by a precise bound when no chessman has been captured. We improve this estimate and show that the number of diagrams is less than 4×10^{37}. To achieve this, we define a graph on the set of diagrams and a notion of class of pawn structure, leading to a method for bounding pawn positions with any number of men on the board.

1 Introduction

The state space of chess is the set of all possible configurations of a chess game. It gives an estimate of the computational complexity of the game. Unfortunately, chess configurations are less easy to define than chess games. Following the definitions given by Francois Labelle [(F. Labelle)], we will call a diagram the contents of the 64 squares. Taking also into account whose turn it is, castling rights, and any en passant square, we define what we call a position.

In a famous paper [(C. Shannon, 1950)], Shannon estimated the number of possible diagrams of the order of $64!/32!(8!)^2(2!)^6 \approx 4.63 \times 10^{42}$. This is, of course, a very rough estimate: it does not consider that some legal diagrams have less than 32 men on the board. Neither does it take into account the fact that some pawns could be promoted. These two factors lead to an underestimation of the number of diagrams. On the other hand, this estimate accounts for a lot of illegal diagrams, most importantly illegal pawn structures.

Chinchalkar [(S. Chinchalkar, 1996)] proved an upper bound on the number of positions of approximately 1.78×10^{46}. Tromp [(J. Tromp)] claims an upper bound of 7.73×10^{45}, stating that "it requires much better documentation to be considered verifiable". More recently, Tromp [(J. Tromp, 2021)] used 10 000 randomly generated positions and checked their legality, yielding an estimated number of legal positions of $(4.48 \pm 0.37) \times 10^{44}$ with 95% confidence level.

As for the diagrams, Steinerberger [(S. Steinerberger, 2015)] improved Shannon's number, giving an upper bound of approximately 1.53×10^{40} for the number of chess diagrams without promotion. For doing so, he used the fact that some men can not occupy any square of the board: each bishop is either light-squared or dark-squared, and pawns can

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not be located on rank 1 or 8. Moreover, he treated separately the case with 32 men on the board, reducing further the number of pawn positions in that case.

In the present work, we drastically reduce the combinatorial complexity of the setup of pawns when 25 to 32 men are on the board. For the cases with less than 25 men on the board, we propose a way to arrange bishops and kings before adding the pawns and the remaining men. These efforts result in an upper bound of less than 4×10^{37} for the number of diagrams without promotion.

The paper is structured as follows. In Section 2 is described a method to compute an upper bound for the number of diagrams. The case with 25 men or more is handled from subsections 2.1 to 2.5. Several tools are combined: we define a partition on the set of diagrams, create a graph on this partition, and define a notion of class of pawn structure. Then is described the possible consequences of any captures on pawn positions. Starting from diagrams with 32 chessmen and decreasing progressively the number of chessmen on the board, we calculate at each step an upper bound on the number of legal pawn and piece positions. In subsection 2.6 is treated the case with 24 men or less. In section 3, detailed results are shown and in section 4 possible improvements and generalizations are proposed.

2 Methods

2.1 A graph on the set of diagrams

Each diagram is associated with a unique quadruplet $\mathcal{P} = (P_w, P_b, p_w, p_b)$ where P_w and P_b are respectively the number of white and black pieces and p_w and p_b are the number of white and black pawns. Note that the term pieces here is excluding pawns. We will use the term men for the physical pieces of the set, including the pawns. As we are interested in the case without promotion, the following inequalities hold: $1 \leq P_w \leq 8$, $1 \leq P_b \leq 8$, $0 \leq p_w \leq 8$ and $0 \leq p_b \leq 8$. A given quadruplet is a subset of the set of diagrams and the family of all quadruplets is a partition of the sets of diagram.

The set of quadruplets can be represented as an oriented graph: each quadruplet is a vertex of the graph and edges are the legal transitions during a game of chess from a quadruplet to another (see Fig.1). These transitions are only due to captures, as we do not take promotions into account. For each edge of the graph, the color and the nature (piece or pawn) of the captured piece is known. Hence we also know the color of the capturing man, but most of the time its nature is unknown, as it can be either a piece or a pawn. On the graph, any quadruplet except the root has at least one and at most four predecessors. For example $(8,8,8,6)$ has just one predecessor which is $(8,8,8,7)$, whereas $(7,6,7,5)$ has four predecessors which are $(8,6,7,5)$, $(7,7,7,5)$, $(7,6,8,5)$ and $(7,6,7,6)$. The edge between $(8,8,8,7)$ and $(8,8,8,6)$ represents the capture of a black pawn. We define the root of the graph as the quadruplet $\mathcal{P}_0 = (8,8,8,8)$. The diagram of the starting position, as well as all the diagrams with 32 men, is associated with this quadruplet.

2.2 Classes of pawn structures

The most important improvement of our method is to compute a better upper bound for the number of legal pawn positions. For that purpose, we define a class of pawn structures as follows: a class is an array of 8 columns. Each column represents a file of the chessboard (first column is the a file,...and last column is the h file) and contains between 0 and 6
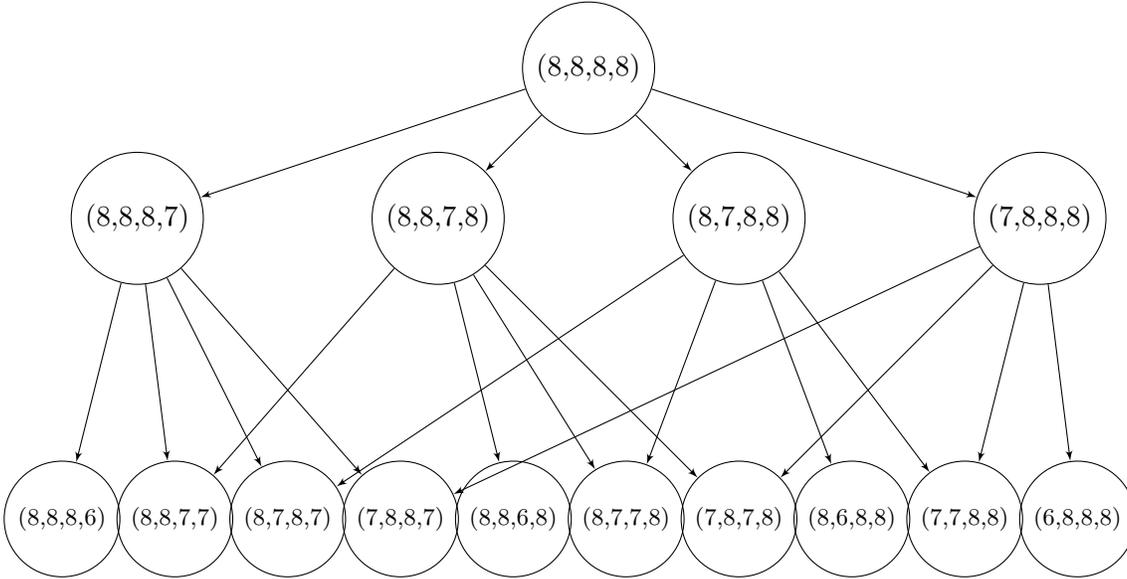
Figure 1: Subgraph for 32, 31 and 30 chessmen.

elements. The length of a column is the number of pawns on the file and the elements of this column are either +1 for white pawns or -1 for black pawns. The order of elements in a column is important: they are written according to their distance to the 7th rank of the chessboard. Thus the element of the first row of a given column is the closest pawn to the 7th rank and the element of the last row is the closest pawn to the second rank. Note that because no pawn can occupy the first and last rank of the board, a column can not contain more than 6 elements.

For each class of pawn structure, it is easy to determine an upper bound for the number of possible positions of black and white pawns. Let us denote by k_i, $i = 1, \ldots, 8$ the number of elements of column i. These k_i pawns can be placed on 6 squares. Since we know the relative position of white and black pawns on each file, the number of different positions of white and black pawns for a given class is equal to $\prod_{i=1}^{8} \binom{6}{k_i}$. A class of pawn structure does not indicate the square occupied by pawns, but only their position relatively to pawns of the same file. Hence, a great number of different pawn positions are encoded in the same class. This makes possible the computation and storage of all the legal classes of a given quadruplet.

The class of pawn structure is modified during a game of chess only in case of a capture or a promotion. Other pawn moves do not change it. For this reason, the class of pawn structure of any legal diagram associated with $\mathcal{P}_0 = (8, 8, 8, 8)$ is the same as the class of the starting position, given in Table 1. Hence, an upper bound for the number of different positions for white and black pawns for this class is $\prod_{i=1}^{8} \binom{6}{2} = 15^8 \approx 2.56e + 9$.

Table 1: The unique class of pawn structure for 32 men diagrams

Col1	Col2	Col3	Col4	Col5	Col6	Col7	Col8
-1	-1	-1	-1	-1	-1	-1	-1
+1	+1	+1	+1	+1	+1	+1	+1

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2.3 Effect of a capture on a class of pawn structure

In order to compute the classes of pawn structures of a given quadruplet, we describe the effect of a capture on a class. Without loss of generality, suppose that the captured man is white. There are four kinds of capture. Here are the consequences of each one:

- when a black piece takes a white piece, the class of pawn structure is unchanged.

- when a black pawn takes a white piece: remove -1 in the array and insert a -1 in an adjacent column (provided this column does not contain 6 elements).

- when a black piece takes a white pawn: remove $+1$ in the array.

- when a black pawn takes a white pawn: for a couple $(-1, +1)$ in adjacent colums, remove -1 from its column and replace $+1$ with -1.

For instance, take a diagram with 32 men. Its class of pawn structure is given in Table 1. Observe the possible classes resulting from the capture of a white pawn. We obtain 8 classes if the pawn is captured by a black piece (one for each possibility of removing $+1$ in the array) and 14 classes if the pawn is captured by a black pawn (one for each couple $(-1, +1)$ in adjacent columns of the array). Now observe what happens with the same initial diagram when a white piece is captured. One class is generated when the capture is made by a black piece. If a black pawn captures a white piece, there are 14 possibilities for choosing adjacent columns and for each one, 3 different places where inserting -1, generating 14×2 different classes after removing duplicates.

2.4 An upper bound for the number of legal pawn positions of a quadruplet

Without promotion, only captures can change the class of pawn structure during a game. Consequently, the list of classes of a given quadruplet $\mathcal{P}$ is inherited from the list of classes of its predecessors in the graph. Moreover, the list of class of the root $\mathcal{P}_0 = (8, 8, 8, 8)$ is known. Note that it contains a unique class. Hence the list of classes of any quadruplet may be computed in a recursive way, starting from the root.

For any quadruplet with 31 to 25 chessmen the following program is executed:

- Compute the predecessors of $\mathcal{P}$.

- For each class of pawn structure of each predecessor $\mathcal{P}'$.

  - Compute and store all the possible classes of pawn structures generated by the captures from $\mathcal{P}'$ to $\mathcal{P}$.

- Remove duplicates from the list of classes.

- Compute the number of pawn positions of each class of $\mathcal{P}$.

- Compute the number of pawn positions $n_\mathcal{P}$ of $\mathcal{P}$.

For example (see subsection 2.3), this algorithm generates 22 classes for $\mathcal{P} = (8, 8, 7, 8)$, resulting in an upper bound of

$$n_\mathcal{P} = 8 \times \binom{6}{1} \times \binom{6}{2}^7 + 14 \times \binom{6}{1} \times \binom{6}{2}^7 \approx 2.26 \times 10^{10},$$

and 29 classes for $\mathcal{P} = (7,8,8,8)$ with an upper bound of

$$n_{\mathcal{P}} = \binom{6}{2}^8 + 28 \times \binom{6}{1} \times \binom{6}{3} \times \binom{6}{2}^6 \approx 4.08 \times 10^{10}.$$

Note that the method takes all possible cases of capture into account for upper bounding the number of pawn positions, but that some of these positions may be illegal, see Section 4 for details.

Symmetry was used in order to reduce the number of computations. For example, the number of pawn positions of (P_w, P_w, p_w, p_b) and (P_w, P_w, p_b, p_w) are equal, the same for (P_w, P_b, p_w, p_b) and (P_b, P_w, p_b, p_w) and for (P_w, P_b, p_w, p_w) and (P_b, P_w, p_w, p_w). The computation time and the number of classes of a quadruplet are quickly increasing from 31 to 25 chessmen. For example $(8,8,8,7)$ has 22 classes, whereas $(6,5,7,7)$ has 12730710. Computing the classes of the former takes less than a second and the latter more than four hours. Four quadruplets represents all the diagrams with 31 men, whereas 60 quadruplets were needed to compute the whole 25 men case. Moreover, the improvement generated by this method of counting pawn positions over the combinatorial method in [(S. Steinerberger, 2015)] is decreasing in the same time: for 32 chessmen, the upper bound on the number of diagrams is divided by 10^7, whereas this ratio is approximately 3 for 25 men. For these reasons this way of counting pawn positions was not used for diagrams with less than 25 chessmen.

2.5   Counting pieces positions

Once the upper bound on the number of pawn positions $n_{\mathcal{P}}$ for a given quadruplet $\mathcal{P} = (P_w, P_b, p_w, p_b)$ is known, pieces are placed on the board. Let $n = 64 - p_w - p_b$ be the number of unoccupied squares. When placing bishops, we take advantage of the fact that two bishops of the same side are placed on squares of different color. Hence the number of positions for these two bishops is bounded by $n^2/4$ when n is even and $(n^2-1)/4$ when n is odd. Let b_w and b_b be respectively the number of white and black bishops. The following algorithm is applied:

- Find all combinations containing P_w white pieces and P_b black pieces

- For each combination

    - Compute the number of possible positions of bishop pair(s), if any, on the $64 - p_w - p_b$ squares.
    - Compute the number of possible positions of remaining bishops, if any.
    - Compute the number of possible positions of remaining pieces on the $64 - p_w - p_b - bw - b_b$ squares.
    - Multiply these numbers to compute the number of piece positions of the combination.

- Add these numbers to compute the number of piece positions $m_{\mathcal{P}}$ of $\mathcal{P}$

The upper bound on the number of diagrams of $\mathcal{P}$ is equal to $n_{\mathcal{P}} \times m_{\mathcal{P}}$.

2.6 Diagrams with less than 25 elements

For diagrams with 2 to 24 chessmen the same method as in [(S. Steinerberger, 2015)] is used, with a slight refinement consisting in not letting kings occupy adjacent squares. First the kings and the bishops are placed on the board, considering how many of the 16 squares of the first and last rank (denote A the set of these squares) they occupy and using the fact that kings may not be adjacent to each other. For any $0 \leq i \leq 6$, let $f_i(b_w, b_b)$ denote the number of ways to place the kings, b_w white bishops and b_b black bishops on the board such that i of these pieces are contained in A. The results are stored in Table 2. Every case has been calculated by combinatorial arguments and checked with computer enumeration of all possibilities.

Some examples of such calculation are given. We begin with only kings on the board. If the white king is in a corner, the black king can occupy 14 squares in A. If the white king is in A but not in a corner, the king can occupy 13 squares in A. For this reason $f_2(0,0) = 4 \times 14 + 12 \times 13 = 212$. We obtain in the same way $f_0(0,0)$ and $f_1(0,0)$ giving the number of legal diagrams with only kings of the board with respectively 0 king and 1 king in A.

Now we describe a more complicated scenario. The proof of the other cases is left to the reader. $f_1(2,2)$ requires 4 bishops on the board and 1 piece in A. Hence there are two cases. In the first case, one king is in A, the other one and the 4 bishops are not in A. As seen above, we first place both kings. There are $f_1(0,0) = 1448$ ways to place them. Consider the color of the square occupied by the king outside A. Two bishops are on squares of this color, yielding 23×22 ways to arrange them, and two bishops are on different color, producing 24×23 cases. Hence, this first case yields $1448 \times 24 \times 23^2 \times 22$ possibilities. In the second case, no king is in A. We infer that we have a bishop in A and other pieces outside A. First we place the kings, there are $f_0(0,0) = 1952$ ways to place them outside A. We consider two subcases. In the first one, the kings are on opposite colors, that is 988 cases, in the second one the kings are on the same color, that is 964 cases. When the kings are on opposite colors, first is chosen which bishop is inside A and on which square it is, yielding 4×8 cases. Next we arrange the bishops outside A, obtaining $23^2 \times 22$ ways of placing them. When the kings are on the same color, we have to distinguish whether the bishop in A is on the same color as the kings or not. If it is, with the same method we obtain $2 \times 8 \times 24 \times 23 \times 22$ ways of placing the four bishops. If the bishop inside A is not on the same color as the kings, we obtain $2 \times 8 \times 24 \times 22 \times 21$ ways of placing the four bishops. Therefore

$$
\begin{aligned}
f_1(2,2) = {} & 1448 \times 24 \times 23^2 \times 22 + 988 \times 4 \times 8 \times 23^2 \times 22 \\
& + 964(2 \times 8 \times 24 \times 23 \times 22 + \times 2 \times 8 \times 24 \times 22 \times 21) = 1130721152.
\end{aligned}
$$

Once Table 2 is computed, we proceed by a rather large case distinction. Every case has a simple combinatorial structure. For the sake of completeness, we sum up the algorithm proposed in [(S. Steinerberger, 2015)], slightly modified as described:

- For all possible values of k kings and bishops contained in A, $0 \leq k \leq 6$,
    - For all possibles values of (b_w, b_b), there is $f_k(b_w, b_b)$ ways of placing two kings, b_w and b_b bishops on the board with precisely k of them in A.
        * There is $48 - (2 + b_w + b_k - k)$ squares on which the p_w white pawns are placed and $48 - (2 + b_w + b_b - k) - p_w$ on which the black pawns are placed.

· There are $62 - b_w - b_k - p_w - p_k$ squares on which to place the remaining men.

The upper bound for each case is simply a product of $f_k(b_w, b_b)$ and binomial coefficients. Then all these numbers are summed over all cases containing the same number of chessmen to generate an upper bound for diagrams with 2 to 24 men on the board. This method yields also a bound when there are 25 men or more on the board but it is far less accurate than the one found by the method based on pawn structures and quadruplets.

3   Results

An upper bound has been obtained for the number of diagrams of every quadruplets containing 25 to 32 chessmen (subsection 2.5) and for diagrams with 2 to 24 chessmen (subsection 2.6). Summing over all possibilities yields that the upper bound for the total number of legal chess diagrams without promotion is equal to

$$3.8521\ldots \times 10^{37}.$$

Some details can be found in Table 3 for diagrams with 23 men or more. We observe that the upper bound for the number of diagrams reaches a maximum when there are 26 or 27 men on the board. The improvement factor, defined as the ratio between the upper bounds of the method described in [(S. Steinerberger, 2015)] and our method is increasing with the number of chessmen.

Computation time is less than 1 second for each quadruplet containing 30 men or more. For 28 men, it is less 1 second for $(8, 8, 8, 4)$, which has 2682 classes of pawn structures ($n_{\mathcal{P}} \approx 1.28 \times 10^{11}$) and 38 seconds for $(7, 7, 7, 7)$, which has 122524 classes ($n_{\mathcal{P}} \approx 1.41 \times 10^{13}$). For 25 men, it is 3 seconds for $(8, 8, 8, 1)$, which has 2512 classes of

Table 2: Counting kings and bishops in first and last rank

$f_0(.,.)$	0	1	2		$f_1(.,.)$	0	1	2
0	1952	89792	1031644		0	1448	99288	1517632
1	89792	4040640	45392336		1	99288	6003920	84799744
2	1031644	45392336	498806704		2	1517632	84799744	1130721152

$f_2(.,.)$	0	1	2		$f_3(.,.)$	0	1	2
0	212	31896	757472		0	0	2968	152320
1	31896	2988432	57608192		1	2968	589008	17763648
2	757472	57608192	978967872		2	152320	17763648	413211008

$f_4(.,.)$	0	1	2		$f_5(.,.)$	0	1	2
0	0	0	10276		0	0	0	0
1	0	38584	2473408		1	0	0	123312
2	10276	2473408	89297152		2	0	123312	9324672

$f_6(.,.)$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	364560

Table 3: Upper bound on the number of diagrams for 25 to 32 men

	32 men	31 men	30 men	29 men	28 men
This method	1.89×10^{33}	1.71×10^{34}	1.64×10^{35}	1.53×10^{36}	5.46×10^{36}
Steinerberger paper	1.89×10^{33}	6.97×10^{39}	4.73×10^{39}	2.29×10^{39}	8.75×10^{38}
Improvement factor	1	408000	28800	1490	160

	27 men	26 men	25 men	24 men	23 men
This method	1.05×10^{37}	1.08×10^{37}	6.14×10^{36}	3.19×10^{36}	5.66×10^{35}
Steinerberger paper	2.78×10^{38}	7.50×10^{37}	1.75×10^{37}	3.54×10^{36}	6.29×10^{35}
Improvement factor	26.4	6.92	2.85	1.11	1.11

pawn structures ($n_{\mathcal{P}} \approx 2.72 \times 10^9$) and 4 hours for $(6, 5, 7, 7)$, which has 12730710 classes ($n_{\mathcal{P}} \approx 3.87 \times 10^{14}$).

The upper bound for diagrams with 2 to 24 men is increasing with the number of men, up to 3.19×10^{36} for 24 men. Summing over all these cases, the upper bound is 3.85×10^{36}. This is approximately 10% better than respectively 3.54×10^{36} and 4.28×10^{36} obtained with the method described in [(S. Steinerberger, 2015)]. Preventing the kings to occupy adjacent squares explains this difference: as an argument, consider that $64 \times 63 = 4032$ is the total number of ways of placing kings on an empty board, but among them there is only 3612 legal diagrams, which is approximately 10% lower.

4 Conclusion

4.1 Further improvements

The argument given in this paper is not far from optimal. It seems difficult to obtain a significant improvement on the number of legal pawn positions: some of the computed classes may be illegal, but they seem to be very few, as none was detected from an examination of 100 randomly chosen classes of 25 men quadruplets. It could also be that inside a given class, some pawn positions are illegal. For example, consider the class of $(8, 6, 7, 8)$ in Table 4. Only a white pawn and two black pieces have been captured. A simple retrograd analysis reveals that the white pawn originally on the c file has taken two pieces and is now in the e file, whereas the black e pawn has taken the white d pawn. For this reason, it is impossible to have a pawn on $e2$ and a pawn on $e3$. Consequently, the number of positions of this class could be bounded by $15^6 \times 14 \times 6$ instead of $15^7 \times 6$ given by our method. We believe that this kind of possible improvement has very little effect because few classes are involved and even in those classes the correction is probably low. The most significant gain would probably be achieved by computing list of classes of pawn structures for 24 men and less, if possible.

Regarding piece positions, our method takes into account light squared and dark squared bishops for any number of men and forbids kings to occupy adjacent squares for 24 or less men on the board. We estimate that we would gain approximately 10% by extending this last rule to 25 men or more, but designing a method achieving that task may prove difficult. As for the bishops, the method we use in subsection 2.3 is not optimal, but the loss is only 13% in the worst case. Of course we also account for a lot of illegal

Table 4: A class of $(8, 6, 7, 8)$

Col1	Col2	Col3	Col4	Col51	Col6	Col7	Col8
-1	-1	-1	-1	1	-1	-1	-1
1	1		-1	1	1	1	1

diagrams but we guess it is a small minority of the total. For these reasons, we conjecture that the number of legal diagrams in the game of chess without promotions is between 10^{37} and 3.5×10^{37}, probably close to 3×10^{37}.

4.2 Possible extensions

One possible continuation of this work would be to take promotions into account. For that purpose, supplementary vertices and edges have to be introduced in the graph described in subsection 2.1. Consider for example vertex $(8, 7, 8, 8)$. Only a black piece is missing on the board. Depending whether a piece or a pawn has taken it, 0, 1 or 2 promotions are possible. This means that this quadruplet has to be partitioned in several vertices, some of them having a possible transition to $(8, 8, 8, 7)$ and to $(9, 7, 7, 8)$ (respectively black and white promotion). It is not obvious to decide whether such a treatment could easily be automated and implemented, in particular regarding storage capacities and computation time. Another idea would be to try to adapt this work to study the number of positions, taking into account castling rights and en passant squares.

4.3 Verifiability

Programs and data have been generated using MatLab on a standard desktop computer and are available upon request.

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