Duality for convex infinite optimization on linear spaces

M. A. Goberna* and M. Volle†

December 7, 2021

Abstract

This note establishes a limiting formula for the conic Lagrangian dual of a convex infinite optimization problem, correcting the classical version of Karney [Math. Programming 27 (1983) 75-82] for convex semi-infinite programs. A reformulation of the convex infinite optimization problem with a single constraint leads to a limiting formula for the corresponding Lagrangian dual, called sup-dual, and also for the primal problem in the case when strong Slater condition holds, which also entails strong sup-duality.

Key words Convex infinite programming · Lagrangian duality· Haar duality· Limiting formulas

Mathematics Subject Classification Primary 90C25; Secondary 49N15 · 46N10

1 Introduction

Given a real linear space X, consider the (algebraic) convex infinite programming (CIP) problem

$$(P)\ \inf_{x\in X} f(x),\ \text{s.t. } f_t(x)\le 0,\ t\in T,$$

where T is an infinite index set and $f, f_t : X \longrightarrow \overline{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$, $t \in T$, are convex proper functions. We denote by

$$E := \bigcap_{t\in T} [f_t \le 0] = \{x \in X : f_t(x) \le 0,\ t \in T\}$$

the feasible set of (P) and define

$$M := \bigcap_{t\in T} \operatorname{dom} f_t \supset E \text{ and } \Delta := M \cap \operatorname{dom} f.$$

*Department of Mathematics, University of Alicante, Alicante, Spain (mgoberna@ua.es). Corresponding author.

†Avignon University, LMA EA 2151, Avignon, France (michel.volle@univ-avignon.fr)

Let $\mathbb{R}_+^{(T)}$ be the positive cone of the space $\mathbb{R}^{(T)}$ of functions $\lambda = (\lambda)_{t\in T} : T \to \mathbb{R}$ whose support $\operatorname{supp}\lambda := \{t \in T : \lambda_t \neq 0\}$ is finite and let $0_{\mathbb{R}^{(T)}}$ be its null element. The ordinary *Lagrangian function* associated to (P) is (see [7], [8], etc.) is $L_0 : X \times \mathbb{R}_+^{(T)} \longrightarrow \overline{\mathbb{R}}$ such that $L_0(x,\lambda) := f(x) + \sum_{t\in T} \lambda_t f_t(x)$, where

$$\sum_{t\in T} \lambda_t f_t(x) := \begin{cases} \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(x), & \text{if } \lambda \neq 0_{\mathbb{R}^{(T)}}, \\ 0, & \text{if } \lambda = 0_{\mathbb{R}^{(T)}}. \end{cases}$$

A slightly different Lagrangian is the one associated with the cone constrained reformulation of (P), that is [14, page 138], the function $L : X \times \mathbb{R}_+^{(T)} \longrightarrow \overline{\mathbb{R}}$ such that

$$L(x,\lambda) := \begin{cases} f(x) + \sum_{t\in T} \lambda_t f_t(x), & \text{if } x \in M,\ \lambda \in \mathbb{R}_+^{(T)}, \\ +\infty, & \text{else.} \end{cases}$$

We call L the *conic Lagrangian* of (P).

For each $x \in X$ we have

$$\sup_{\lambda\in\mathbb{R}_+^{(T)}} L_0(x,\lambda) = \sup_{\lambda\in\mathbb{R}_+^{(T)}} L(x,\lambda) = f(x) + \delta_E(x),$$

where δ_E is the indicator of E, that is, $\delta_E(x) = 0$ if $x \in E$ and $\delta_E(x) = +\infty$ otherwise. Consequently,

$$\inf_{x\in X} \sup_{\lambda\in\mathbb{R}_+^{(T)}} L_0(x,\lambda) = \inf_{x\in X} \sup_{\lambda\in\mathbb{R}_+^{(T)}} L(x,\lambda) = \inf(P).$$

The *ordinary* and *conic-Lagrangian dual problems* of (P) read, respectively,

$$(D_0) \quad \sup_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in X} \left(f(x) + \sum_{t\in T} \lambda_t f_t(x) \right),$$

and

$$(D) \quad \sup_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in M} \left(f(x) + \sum_{t\in T} \lambda_t f_t(x) \right),$$

and one has

$$\sup(D_0) \leq \sup(D) \leq \inf(P). \tag{1.1}$$

Note that, if $\operatorname{dom} f \subset M$, then $\sup(D_0) = \sup(D)$. This is, in particular, the case when the functions f_t, $t \in T$, are real-valued. But it may happen that $\sup(D_0) < \sup(D)$ even if T is finite and Slater condition holds. This is the case in the next example.

Example 1.1 *Consider* $X = \mathbb{R}^2$, $T = \{1\}$, $f(x_1,x_2) = e^{x_2}$, *and*

$$f_1(x_1,x_2) = \begin{cases} x_1, & \text{if } x_2 \geq 0, \\ +\infty, & \text{if } x_2 < 0. \end{cases}$$

We then have

$$\max(D_0) = 0 < 1 = \max(D) = \min(P).$$

Duffin [5] observed that a positive duality gap might occur when one considers the ordinary Lagrangian dual (D_0) of (P). The same happens when (D_0) is replaced by (D) even though, according to (1.1), the gap may be smaller. Different ways have been proposed to close the duality gap, e.g., by adding a linear perturbation to the saddle function $f + \sum_{t\in T} \lambda_t f_t$, and sending it to zero in the limit [5]. Blair, Duffin and Jeroslow [1] used the conjugate duality theory to extend the limiting phenomena to the general minimax setting. Pomerol [12] showed that it was possible to obtain infisup theorems, including that of [1], by using a slightly more general form of the duality theory. In turn, Karney and Morley [9] proved that, when $X = \mathbb{R}^n$, either the convex semi-infinite programming (CSIP in brief) problem (P) satisfies some recession condition guaranteeing a zero duality gap or there exists $d \in \mathbb{R}^n \diagdown \{0_n\}$ such that the problem

$$(P_\varepsilon) \quad \inf_{x\in X} f(x) + \varepsilon \langle d, x\rangle, \text{ s.t. } f_t(x) \leq 0, \ t \in T,$$

satisfies the mentioned recession condition for $\varepsilon > 0$ sufficiently small, with (P_ε) enjoying strong duality, and $\inf (P) = \lim_{\varepsilon\downarrow 0} (P_\varepsilon)$. The theory developed in [9] subsumed the CSIP versions of some results on limiting Lagrangians in [2] and [6]. Three years before, Karney gave, in the CSIP setting, a limiting formula for the dual problem (D_0):

$$\sup(D_0) = \lim_{\varepsilon\downarrow 0} \inf \{f(x) : \ f_t(x) \leq \varepsilon, \ t \in T\}. \tag{1.2}$$

According to [8, Proposition 3.1], this formula comes from [13, Theorem 7] and [2, Corollary 2], and does not require any constraint qualification (other than $E \neq \emptyset$, or something stronger as $E \cap \operatorname{dom} f \neq \emptyset$, $E \subset \operatorname{cl} \operatorname{dom} f$, ...). The next example shows that [8, Proposition 3.1] fails even in linear semi-infinite programming, where $\operatorname{dom} f = X = \mathbb{R}^n$, while [13, Theorem 7] and [2, Corollary 2] hold.

Example 1.2 *Consider the following optimization problem, with $T = \mathbb{N}$:*

$$\begin{array}{llll} (P) & \inf_{x\in\mathbb{R}^2} & x_2 & \\ & s.t. & x_1 \leq 0, & (t=1) \\ & & -x_2 \leq 1, & (t=2) \\ & & t^{-1}x_1 - x_2 \leq 0, & t = 3,4,... \end{array}$$

Its dual problem (D_0), that is also (D), is equivalent to the Haar dual (see, e.g., [7])

$$\begin{array}{ll} \sup_{\lambda\in\mathbb{R}_+^{(\mathbb{N})}} & -\lambda_2 \\ s.t. & \lambda_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{t\geq 3} \lambda_t \begin{pmatrix} -t^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{array}$$

whose unique feasible solution is $\lambda \in \mathbb{R}_+^{(\mathbb{N})}$ such that $\lambda_2 = 1$ and $\lambda_t = 0$ for $t \neq 2$. So, $\max (D_0) = -1$ *while* $E = \{(x_1, x_2) : x_1 \leq 0, x_2 \geq 0\}$, *so that* $\min (P) = 0$. *On the other hand, given $\varepsilon > 0$,*

$$\left\{x \in \mathbb{R}^2 : \ f_t(x) \leq \varepsilon, \ t \in \mathbb{N}\right\} = \left\{x \in \mathbb{R}^2 : x_1 \leq \varepsilon, x_2 \geq -\varepsilon, \frac{x_1}{3} - x_2 \leq \varepsilon\right\},$$

so that

$$\min\{x_2 : f_t(x) \leq \varepsilon,\ t \in \mathbb{N}\} = -\varepsilon$$

is attained at $\{(x_1, -\varepsilon) : x_1 \leq 0\}$. *Hence,*

$$\max(D_0) = -1 < 0 = \lim_{\varepsilon \downarrow 0} \min\{x_2 : f_t(x) \leq \varepsilon,\ t \in \mathbb{N}\}.$$

From [8, Proposition 3.1] Karney obtained, following the suggestion of an unknown referee, the reverse strong duality theorem [8, Theorem 3.2] guaranteeing zero dualily gap with primal attainment, i.e.,

$$\min(P) = \sup(D_0),$$

under some recession condition. However, he asserted in [8, Section 5] that he had two (longer) unpublished proofs. In either case, his result has been recently proved from a new strong duality theorem for CIP (see [4, Corollary 3.2 and Remark 3.2]).

In this note we show in a simpler way, for general CIP problems, that, under the strong Slater condition

$$\exists \alpha > 0, \exists a \in \operatorname{dom} f : \quad f_t(a) \leq -\alpha,\ \forall t \in T,$$

(1.2) entails that zero duality gap holds:

$$\sup(D_0) = \inf(P).$$

This duality theorem is obtained by studying the Lagrangian dual (D_1) associated with the representation of E by a single constraint (the so-called sup-function). Section 2 (resp. Section 3) provides a limiting formula for $\sup(D)$ (resp. $\sup(D_1)$). Under the strong Slater condition, the limiting formula for $\sup(D_1)$ also holds for $\inf(P)$ together with the strong duality theorem $\inf(P) = \max(D_1)$.

2 Conic-Lagrangian duality

Problem (D) receives a perturbational interpretation (see [3], [14], etc.) in terms of the *ordinary value function* $v : \mathbb{R}^T \longrightarrow \overline{\mathbb{R}}$ associated with (P) defined by

$$v(y) := \inf\{f(x) : f_t(x) \leq y_t, t \in T\}, \forall y = (y_t)_{t \in T} \in \mathbb{R}^T.$$

Let us make this approach explicit. The linear space $Y := \mathbb{R}^T$, equipped with the product topology, is a locally convex Hausdorff topological vector space whose topological dual is $\mathbb{R}^{(T)}$ via the bilinear pairing

$$\langle \cdot, \cdot \rangle : Y \times \mathbb{R}^{(T)} \longrightarrow \mathbb{R} \text{ such that } \langle y, \lambda \rangle = \sum_{t \in T} \lambda_t y_t.$$

The Fenchel conjugate of v is (see [3], [14], etc.)

$$-v^{*}(-\lambda)=\begin{cases}\inf_{x\in\Delta}\left(f(x)+\sum_{t\in T}\lambda_t f_t(x)\right), & \text{if } \Delta\neq\emptyset \text{ and } \lambda\in\mathbb{R}_+^{(T)},\\ -\infty, & \text{if } \Delta=\emptyset \text{ or } \lambda\in\mathbb{R}^{(T)}\diagdown\mathbb{R}_+^{(T)}.\end{cases}\tag{2.1}$$

If $\Delta\neq\emptyset$ we the have

$$\begin{aligned} v^{**}(0_Y) &= \sup_{\lambda\in\mathbb{R}^{(T)}}-v^{*}(\lambda)=\sup_{\lambda\in\mathbb{R}^{(T)}}-v^{*}(-\lambda)=\sup_{\lambda\in\mathbb{R}_+^{(T)}}-v^{*}(-\lambda)\\ &= \sup_{\lambda\in\mathbb{R}_+^{(T)}}\inf_{x\in\Delta}\left(f(x)+\textstyle\sum_{t\in T}\lambda_t f_t(x)\right)=\sup(D).\end{aligned}$$

Note that, if $\Delta=\emptyset$ we have $\operatorname{dom} v=\emptyset$ and $v^{**}(0_Y)=+\infty=\sup(D)$. Therefore, in all cases we have

$$\sup(D)=v^{**}(0_Y)\leq\overline{v}(0_Y)\leq v(0_Y)=\inf(P),\tag{2.2}$$

where $\overline{v}$ is the lower semicontinuous (lsc in brief) hull of v for the product topology on $Y=\mathbb{R}^T$. A neighborhood basis of the origin 0_Y is furnished by the family

$$\left\{V_\varepsilon^H:\varepsilon>0,H\in\mathcal{F}(T)\right\},$$

where $\mathcal{F}(T)$ is the class of non-empty finite subsets of T, and

$$V_\varepsilon^H:=\{y\in Y:|y_t|\leq\varepsilon,t\in H\}.$$

We now give a general explicit formula for $\overline{v}(0_Y)$:

Lemma 2.1 $\overline{v}(0_Y)=\sup\limits_{\varepsilon>0,H\in\mathcal{F}(T)}\inf\limits_{x\in M}\{f(x):f_t(x)\leq\varepsilon,t\in H\}.$

Proof For each $\varepsilon>0$ and $H\in\mathcal{F}(T)$ one has

$$\begin{aligned}\inf_{y\in V_\varepsilon^H}v(y) &= \inf\{f(x):f_t(x)\leq y_t,t\in T;|y_t|\leq\varepsilon,t\in H\}\\ &= \inf\{f(x):f_t(x)\leq\varepsilon,t\in H;f_t(x)<+\infty,t\notin H\}\\ &= \inf_{x\in M}\{f(x):f_t(x)\leq\varepsilon,t\in H\}.\end{aligned}$$

Since $\overline{v}(0_Y)=\sup\limits_{\varepsilon>0,H\in\mathcal{F}(T)}\inf\limits_{y\in V_\varepsilon^H}v(y)$, we are done. □

Remark 2.1 *From Lemma 2.1 one gets*

$$\overline{v}(0_Y)\leq\lim_{\varepsilon\downarrow 0}\inf\{f(x):f_t(x)\leq\varepsilon,t\in T\}.$$

Remark 2.2 *In the case when the index set T is finite, the formula provided by Lemma 2.1 can be simplified as follows:*

$$\overline{v}\left(0_Y\right) = \lim_{\varepsilon\downarrow 0} \inf\left\{f\left(x\right) : f_t\left(x\right) \leq \varepsilon, t \in T\right\}.$$

In such a case we also have $M = \bigcap_{t\in T} \operatorname{dom} f_t$ and

$$v^{**}\left(0_Y\right) = \sup_{\lambda\in\mathbb{R}_+^T} \inf_{x\in M} \left(f\left(x\right) + \sum_{t\in T} \lambda_t f_t\left(x\right)\right).$$

Proposition 2.1 (Limiting formula for $\sup(D)$) *Assume either $\overline{v}\left(0_Y\right) \neq +\infty$ or $\sup(D) \neq -\infty$. Then we have*

$$\sup(D) = \sup_{\varepsilon>0, H\in\mathcal{F}(T)} \inf_{x\in M} \left\{f\left(x\right) : f_t\left(x\right) \leq \varepsilon, t \in H\right\}.$$

Proof We know that $\sup(D) = v^{**}\left(0_Y\right)$ (see (2.2)). Since the functions f and f_t, $t \in T$, are convex, the value function v is convex, too. By [2, Proposition 1], we then have $\sup(D) = \overline{v}\left(0_Y\right)$ and Lemma 2.1 concludes the proof. $\square$

Remark 2.3 *Condition $\overline{v}\left(0_Y\right) \neq +\infty$ is in particular satisfied if $\inf(P) \neq +\infty$, that is $E \cap \operatorname{dom} f \neq \emptyset$.*
Condition $\sup(D) \neq -\infty$ is satisfied if and only if there exists $\lambda \in \mathbb{R}_+^{(T)}$ and $r \in \mathbb{R}$ such that

$$x \in M \Longrightarrow f\left(x\right) + \sum_{t\in T} \lambda_t f_t\left(x\right) \geq r.$$

Remark 2.4 *By (1.1), (2.1) and (2.2), we have*

$$\sup(D_0) \leq \sup(D) \leq \lim_{\varepsilon\downarrow 0} \inf\left\{f\left(x\right) : f_t\left(x\right) \leq \varepsilon, t \in T\right\}.$$

In [8, Proposition 3.1] it is claimed that for $X = \mathbb{R}^n$, f and f_t, $t \in T$, are proper, lsc and convex, and $E \neq \emptyset$, it holds that

$$\sup(D_0) = \lim_{\varepsilon\downarrow 0} \inf\left\{f\left(x\right) : f_t\left(x\right) \leq \varepsilon, t \in T\right\}.$$

To the best of our knowledge, this fact has not been proved anywhere. We prove in Proposition 3.2 below an exact formula for its right-hand side.

3 Sup-Lagrangian duality

Let $h := \sup_{t\in T} f_t$ be the *sup-function* of (P) which allows to represent its feasible set E with a single constraint. We associate with (P) another Lagrangian $L_1 : X \times \mathbb{R}_+ \longrightarrow \overline{\mathbb{R}}$, called *sup-Lagrangian,* such that

$$L_1(x,s) := \begin{cases} f(x) + sh(x), & \text{if } x \in \Delta_1 := \operatorname{dom} f \cap \operatorname{dom} h \text{ and } s \geq 0, \\ +\infty, & \text{else.} \end{cases}$$

Note that $\Delta_1 \subset \Delta$. For each $x \in X$ we have

$$\sup_{s\geq 0} L_1(x,s) = f(x) + \delta_E(x),$$

and

$$\inf_{x\in X} \sup_{s\geq 0} L_1(x,s) = \inf(P).$$

The corresponding Lagrangian dual problem, say *sup-dual problem*, reads

$$(D_1) \ \sup_{s\geq 0} \inf_{x\in\Delta_1} (f(x) + sh(x)).$$

Let us introduce the *sup-value function* $v_1 : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ associated with (P) via L_1, namely,

$$v_1(r) := \inf\{f(x) : h(x) \leq r\}, \ r \in \mathbb{R},$$

which is non-increasing and satisfies

$$\overline{v}_1(0) = \lim_{\varepsilon\downarrow 0} v_1(\varepsilon) = \lim_{\varepsilon\downarrow 0} \inf\{f(x) : f_t(x) \leq \varepsilon, t \in T\}. \tag{3.1}$$

Lemma 3.1 $\sup(D) \leq \sup(D_1) \leq \inf(P)$.

Proof Let us prove the first inequality (the second being obvious). Given $\lambda \in \mathbb{R}_+^{(T)}$, one has to check that

$$\inf_{x\in\Delta} \left(f(x) + \sum_{t\in T} \lambda_t f_t(x) \right) \leq \sup(D_1).$$

If $\operatorname{supp} \lambda = \emptyset$, then

$$\inf_{x\in\Delta} \left(f(x) + \sum_{t\in T} \lambda_t f_t(x) \right) = \inf_{x\in\Delta} f \leq \inf_{x\in\Delta_1} f \leq \sup(D_1)$$

and we are done.

If $\operatorname{supp}\lambda \neq \emptyset$, one has, for $s = \sum_{t \in T} \lambda_t$,

$$\begin{aligned} \sup(D_1) &\geq \inf_{x \in \Delta_1} (f(x) + sh(x)) \\ &\geq \inf_{x \in \Delta_1} \left(f(x) + s \sum_{t \in T} \tfrac{\lambda_t}{s} f_t(x)\right) \\ &\geq \inf_{x \in \Delta_1} \left(f(x) + \sum_{t \in T} \lambda_t f_t(x)\right) \\ &\geq \inf_{x \in \Delta} \left(f(x) + \sum_{t \in T} \lambda_t f_t(x)\right). \end{aligned}$$

□

Proposition 3.1 (Limiting formula for $\sup(D_1)$**)** *Assume that either* $\overline{v}_1(0) \neq +\infty$ *or* $\sup(D_1) \neq -\infty$. *Then we have*

$$\sup(D_1) = \lim_{\varepsilon \downarrow 0} \inf \{f(x) : f_t(x) \leq \varepsilon, t \in T\}.$$

Proof By (3.1), the right-hand side of (3.1) coincides with $\overline{v}_1(0)$. By definition of v_1 we have (as for v), $v_1^{**}(0) = \sup(D_1)$. Since v_1 is convex and either $\overline{v}_1(0) \neq +\infty$ or $v_1^{**}(0) \neq -\infty$, we then have, by [2, Proposition 1], $\sup(D_1) = \overline{v}_1(0)$ and we are done. □

Proposition 3.2 (Limiting formula for $\inf(P)$**)** *Assume that the strong Slater condition*

$$\exists \alpha > 0,\ \exists a \in \operatorname{dom} f : \quad f_t(a) \leq -\alpha,\ \forall t \in T, \tag{3.2}$$

holds. Then we have

$$\inf(P) = \max_{s \geq 0} \inf_{x \in \Delta_1} (f(x) + sh(x)) = \lim_{\varepsilon \downarrow 0} \inf \{f(x) : f_t(x) \leq \varepsilon, t \in T\}. \tag{3.3}$$

Proof By definition of h we have

$$\inf(P) = \inf \{f(x) : h(x) \leq 0\}.$$

Note that (3.2) amounts to the usual Slater condition relative to h :

$$\exists a \in \operatorname{dom} f : \quad h(a) < 0.$$

Since the functions f and h are convex, we then have (see, e.g., [10, Lemma 1])

$$\inf(P) = \max_{s \geq 0} \inf_{x \in \Delta_1} (f(x) + sh(x)) = \max(D_1).$$

By (3.2) we have $\overline{v}_1(0) \leq v_1(0) < +\infty$. By Proposition 3.1 it follows that

$$\sup(D_1) = \lim_{\varepsilon \downarrow 0} \inf \{f(x) : f_t(x) \leq \varepsilon, t \in T\}$$

and we are done. □

Let us revisit Example 1.2, where (3.3) fails. Any candidate a to be strong Slater point is feasible. Let a be a feasible solution of (P). Then $a = (a_1, 0)$, with $a_1 \leq 0$, and $h(a) \geq \sup\{t^{-1}a_1 : t = 3, 4, ...\} = 0$. Thus, $h(a) = 0$ and the strong Slater constraint qualification (3.2) fails. However, by Proposition 3.1, we have

$$\sup(D_1) = \lim_{\varepsilon \downarrow 0} \inf\{f(x) : h(x) \leq \varepsilon\} = \lim_{\varepsilon \downarrow 0} -\varepsilon = 0$$

and, finally,

$$\begin{aligned} -1 = \sup(D_0) = \sup(D) < \sup(D_1) = 0 = \min(P) \\ = \inf\{f(x) : h(x) = 0\} = \lim_{\varepsilon \downarrow 0} \inf\{f(x) : h(x) \leq \varepsilon\}. \end{aligned}$$

Remark 3.1 *In the case when T is finite, condition (3.2) reads*

$$\exists a \in \operatorname{dom} f : \quad f_t(a) < 0, \ \forall t \in T,$$

that is the familiar Slater constraint qualification. One also has $\Delta_1 = \left(\bigcap_{t \in T} \operatorname{dom} f_t \right) \cap \operatorname{dom} f$ *and, by Proposition 3.2, there exists $\overline{s} \geq 0$ such that*

$$\inf(P) = \inf_{x \in \Delta_1} (f(x) + \overline{s}h(x)) = \inf_{x \in \Delta_1} \sup_{\nu \in S_T} \left(f(x) + \overline{s} \sum_{t \in T} \nu_t f_t(x) \right),$$

where $S_T = \left\{ \nu \in \mathbb{R}_+^T : \sum_{t \in T} \nu_t = 1 \right\}$ *is the unit simplex in $\mathbb{R}^T$. By the minimax theorem [14, Theorem 2.10.1], with $A = S_T$ and $B = \Delta_1$, there exists $\overline{\nu} \in S_T$ such that*

$$\inf(P) = \inf_{x \in \Delta_1} \left(f(x) + \overline{s} \sum_{t \in T} \overline{\nu}_t f_t(x) \right) \leq \sup(D) \leq \inf(P)$$

and, consequently, $\inf(P) = \max(D)$, *which is the strong duality theorem [14, Theorem 2.9.3] without assuming a topological structure on the basic linear space X (see also [11, Remark 8]).*

Concerning Example 1.1, let us note that

$$\max(D_0) = 0 < 1 = \max(D) = \lim_{\varepsilon \downarrow 0} \inf\{f(x) : f_1(x) \leq \varepsilon\} = \min(P),$$

which also contradicts [8, Proposition 3.1].

Acknowledgement This research was partially supported by Ministerio de Ciencia, Innovación y Universidades (MCIU), Agencia Estatal de Investigación (AEI), and European Regional Development Fund (ERDF), Project PGC2018-097960-B-C22.

References

[1] Blair, C.E., Duffin, R.J., Jeroslow, R.G.: A limiting infisup theorem. J Optim Theory Appl **37**, 163-175 (1982)

[2] Borwein, J.M.: A note on perfect duality and limiting Lagrangeans. Math. Programming **18**, 330-337 (1980)

[3] Boţ, R.I.: Conjugate Duality in Convex Optimization. Springer-Verlag, Berlin/Heidelberg (2010)

[4] Dinh, N., Goberna, M.A., López, M.A., Volle, M.: Relaxed Lagrangian duality in convex infinite optimization: reverse strong duality and optimality. Preprint. Available at http://arxiv.org/abs/2106.09299

[5] Duffin, R.J.: Convex analysis treated by linear programming. Math. Programming **4**, 125-143 (1973)

[6] Duffin, R.J., Jeroslow, R.G.: The Limiting Lagrangian. Georgia Institute of Technology, Management Science Technical Reports No. MS-79-13 (1979)

[7] Goberna, M.A., López, M.A.: Linear Semi-Infinite Optimization. J. Wiley, Chichester, U.K., (1998)

[8] Karney, D.F.: A duality theorem for semi-infinite convex programs and their finite subprograms. Math. Programming **27**, 75-82 (1983)

[9] Karney, D.F., Morley, T.D.: Limiting Lagrangians: A primal approach. J Optim. Theory Appl. **48**, 163-174 (1986).

[10] Lemaire, B., Volle, M.: Duality in DC programming. Generalized convexity, generalized monotonicity: recent results (Luminy, 1996), 331-345, Nonconvex Optim. Appl. **27**, Kluwer, Dordrecht (1998)

[11] Luc, D.T., Volle, M.: Algebraic approach to duality in optimization and applications. Set-Valued Var. Anal. **29**, 661-681 (2021)

[12] Pomerol, J.-Ch.: A note on limiting infisup theorems. Math. Programming **30**, 238-241 (1984)

[13] Rockafellar, R.T.: Conjugate Duality and Optimization. SIAM, Philadelphia, P.A. (1974)

[14] Zălinescu, C.: Convex analysis in general vector spaces. World Scientific, River Edge, N.J. (2002)