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Abstract: We consider a smooth control system that is subject to loss of control in the sense that the state space is partitioned into several disjoint regions and, in each region, either the system can be controlled, as usual, in a permanent way (that is, one can change the value of the control at any real time), or, on the contrary, the control has to remain constant from the entry time into the region until the exit time. The latter case corresponds to a non-control region. The objective of this paper is to state the necessary optimality conditions for a Mayer optimal control problem in such a setting of loss of control. Our main result is based on a hybrid maximum principle that takes into account a regionally switching parameter.

Keywords: Optimal control, non-control regions, Pontryagin maximum principle.

1. INTRODUCTION

Optimal control theory has been successfully applied to solve optimal control problems arising in various areas such as engineering, biology, aeronautics, aerospace, etc. The two fundamental principles are namely the Pontryagin maximum principle (see Pontryagin et al. (1964)) and the Hamilton-Jacobi-Bellman equation (see Bellman (1957)) that have been developed at the end of the fifties. Both methods allow to synthesize optimal control strategies that are implemented in real world experiments. Nowadays there is still a need of developing further optimality conditions to cover various situations encountered in practice.

The usual tools of optimal control theory presuppose that it is possible to modify the value of the control at any real time. In this paper, we are interested in studying optimal control problems for which it is not possible to act on the control system everywhere in a permanent way, that is, according to the position of the system in the state space, a loss of control can be suffered. Such a situation typically occurs in aerospace when a spacecraft enters into a shadow zone (see Geffroy and Epenoy (1997); Trélat (2012) and references therein). Let us also mention time crisis problems in the context of viability theory (see Aubin et al. (2011)). Such a problem consists in minimizing the total time spent by a control system outside a given set K (see Bayen and Pfeiffer (2020)). Depending on the application model, one important issue is to consider situations where a practitioner cannot modify the value of the control in the set K, or in its complement (such as in population models, see Bayen and Rapaport (2019)).

To model the loss of control, we consider a partition of the state space into several disjoint regions (in the spirit of Haberkorn and Trélat (2011)) and we suppose that each region can be of type C or NC (where C and NC stand for *control region* and *non-control region*). In a region of type C, the system can be controlled, as usual, in a permanent way (that is, one can change the value of the control at any real time), whereas, in a region of type NC, the control is frozen at the entry time into the region until the exit time. In the latter case, the constant value of the control can be chosen (as a parameter), but the control has to remain constant as long as the system belongs to the region.

In this paper, we consider a Mayer optimal control problem in which the control system is governed by a smooth dynamics and is subject to loss of control as explained above. Our objective is to write the corresponding necessary optimality conditions in a Pontryagin form. As far as we know, such a framework has never been considered in the literature. Let us emphasize that the Pontryagin maximum principle (PMP in short) has been adapted to numerous situations, in particular whenever the problem depends on additional parameters that remain constant over the whole time interval. More generally, piecewise constant controls (called *sampled-data controls*) have been considered in Bourdin and Trélat (2016); Bourdin and Dhar (2019, 2020) in which the control remains constant over time subintervals (respecting fixed or free time partitions). However, note that these settings are not adapted to the present work because, here, the time subintervals over which the control has to remain constant depend on the state position.

Our strategy is as follows. We embed our problem into a hybrid optimal control problem including a parameter that is constant on each region, but can change its value from one region to another. We speak of a *regionally switching parameter*. The corresponding necessary optimality conditions in this more general setting have been derived in a subsequent work (see Bayen et al. (2022)). Let us emphasize that, because of the presence of the regionally switching parameter, one cannot apply the hybrid maximum principles available in the literature (such as in Sussmann (1999); Riedinger et al. (2003); Garavello and Piccoli (2005); Caines et al. (2006); Clarke and Vinter (1989); Haberkorn and Trélat (2011); Clarke (2013); Barles et al. (2018)). Note that our methodology also differs from Dmitruk and Kaganovich (2008) which provides a hybrid maximum principle adapted to a dynamics subject to changes at some (fixed or free) instants that are independent of the state position. The techniques we use in Bayen et al. (2022) rely on a thorough sensitivity analysis in the context of differential equations that is extended to hybrid systems. Such an analysis allows us to compute variation vectors along the hybrid trajectory when considering perturbations of the parameter and of the control in a given region. A crucial hypothesis is the transversality of the optimal trajectory at each crossing time in order to obtain perturbed trajectories possessing the same structure as the nominal one.

The paper is organized as follows. In Section 2, we introduce the framework and state our main result (Theorem 1) showing that:

- in a region of type C, the usual Hamiltonian maximization condition holds true;
- in a region of type NC, an averaged Hamiltonian gradient condition in the spirit of Bourdin and Trélat (2016); Bourdin and Dhar (2019, 2020) is fulfilled.

To keep this work concise, we do not include the proof of the necessary optimality conditions, but we refer to Bayen et al. (2022) for details. In Section 3, a simple example is developed for illustration. Finally, the paper is concluded with perspectives for further research works in Section 4.

2. MAIN RESULT

We first introduce some notations. Throughout this paper, let m, $n \in \mathbb{N}^*$ be two fixed positive integers and let $T > 0$ be a fixed positive real number. In the sequel, $\langle \cdot, \cdot \rangle$ denotes the inner product over $\mathbb{R}^n$ and $\overline{B}[x, r]$ denotes the closed ball in $\mathbb{R}^n$ of center $x \in \mathbb{R}^n$ and of radius $r > 0$. Then we denote by $\overline{S}$ and ∂S the closure and the boundary of any subset S of $\mathbb{R}^n$. Finally, we denote by $\mathrm{AC}([0, T], \mathbb{R}^n)$ (resp. $\mathrm{L}^\infty([0, T], \mathbb{R}^m)$) the space of absolutely continuous functions (resp. essentially bounded functions) defined on $[0, T]$ with values in $\mathbb{R}^n$ (resp. $\mathbb{R}^m$).

2.1 Mayer optimal control problem with non-control regions

Throughout this paper we consider a partition of $\mathbb{R}^n$ given by

$$\mathbb{R}^n = \bigcup_{j \in \mathcal{J}} \overline{X_j},$$

where $\mathcal{J}$ is a (possibly infinite) family of indexes and the nonempty connected open subsets X_j (called *regions*) are disjoint. We assume that each region is either of type C, either of type NC (see Introduction for details), and thus, for all $j \in \mathcal{J}$, we introduce

$$q_j := \begin{cases} 1 & \text{if } X_j \text{ is a control region,} \\ 0 & \text{if } X_j \text{ is a non-control region.} \end{cases}$$

Now we introduce the control system

$$\begin{cases} (x, u) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m), \\ \dot{x}(t) = f(x(t), u(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_{\mathrm{init}}, \\ u \text{ is constant in non-control regions,} \end{cases} \tag{CS}$$

where the initial condition x_{init} is fixed with $x_{\mathrm{init}} \in X_{j_1}$ for some $j_1 \in \mathcal{J}$, and the dynamics $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1. As usual in the literature, x is called *state* (or *trajectory*) and u is called *control*. We now give a precise definition of a solution to (CS).

Definition 1. We say that a pair $(x, u) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m)$ is a *solution* to (CS) if the four following conditions are satisfied:

(1) There exists a partition $\mathbb{T} = \{t_k^c\}_{k=0,\dots,N}$ of $[0, T]$, with $N \in \mathbb{N}^*$, such that, for all $k \in \{1, \dots, N\}$, there exists $j(k) \in \mathcal{J}$ such that

$$\forall t \in (t_{k-1}^c, t_k^c), \;\; x(t) \in X_{j(k)},$$

with $j(k) \neq j(k-1)$ for all $k \in \{2, \dots, N\}$. In the sequel, the times t_k^c, for $k \in \{1, \dots, N-1\}$, are called *crossing times* and correspond to the instants at which the trajectory x goes from one region to another (in particular $x(t_k^c)$ belongs to the interface $\partial X_{j(k)} \cap \partial X_{j(k+1)}$).

(2) It holds that $x(0) \in X_{j(1)}$ and $x(T) \in X_{j(N)}$.

(3) The state equation $\dot{x}(t) = f(x(t), u(t))$ is satisfied for almost every $t \in [0, T]$ and $x(0) = x_{\mathrm{init}}$ (and thus $j(1) = j_1$).

(4) For all $k \in \{1, \dots, N\}$ such that $q_{j(k)} = 0$, the control u is constant over (t_{k-1}^c, t_k^c) (the constant value being denoted by u_k in the sequel).

Our objective in the present work is to derive first-order necessary optimality conditions (in a PMP form) for the Mayer optimal control problem with non-control regions given by

$$\begin{aligned} &\text{minimize} \quad \phi(x(T)), \\ &\text{subject to } (x, u) \text{ solution to (CS)}, \\ &\qquad\qquad u(t) \in \mathrm{U} \text{ a.e. } t \in [0, T], \end{aligned} \tag{OCP}$$

where the Mayer cost function $\phi : \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and the control constraint set U is a nonempty closed convex subset of $\mathbb{R}^m$.

2.2 Regular solution and necessary optimality conditions

Our main result is based on some regularity assumptions concerning the transverse behavior of the optimal trajectory at the interfaces between regions. The precise hypotheses are provided in the next definition and are standard (see, *e.g.*, Haberkorn and Trélat (2011); Bayen and Pfeiffer (2020)).

Definition 2. Consider a solution (x, u) to (CS) and the notations introduced in Definition 1. Set $\alpha := \frac{1}{3} \min_{k=1,\dots,N} |t_k - t_{k-1}| > 0$. We say that (x, u) is *regular* if there exist $\delta \in (0, \alpha)$ and $\nu > 0$ such that:

(1) At each crossing time t_k^c, the control u is continuous over $[t_k^c - \delta, t_k^c)$ and over $(t_k^c, t_k^c + \delta]$, and admits left and right limits denoted by $u^-(t_k^c)$ and $u^+(t_k^c)$.
(2) At each crossing time t_k^c, there exists a C^1 function $F_k : \overline{B}[x(t_k^c), \nu] \to \mathbb{R}$ such that

$$\begin{cases} y \in X_{j(k)} \Leftrightarrow F_k(y) < 0, \\ y \in \partial X_{j(k)} \cap \partial X_{j(k+1)} \Leftrightarrow F_k(y) = 0, \\ y \in X_{j(k+1)} \Leftrightarrow F_k(y) > 0, \end{cases}$$

for all $y \in \overline{B}[x(t_k^c), \nu]$.
(3) At each crossing time t_k^c, the *transverse conditions* depicted in Figure 1 and given by

$$\langle \nabla F_k(x(t_k^c)), (f)^-(t_k^c) \rangle > 0,$$
$$\langle \nabla F_k(x(t_k^c)), (f)^+(t_k^c) \rangle > 0,$$

are fulfilled, where $(f)^\pm(t_k^c) := f(x(t_k^c), u^\pm(t_k^c))$.

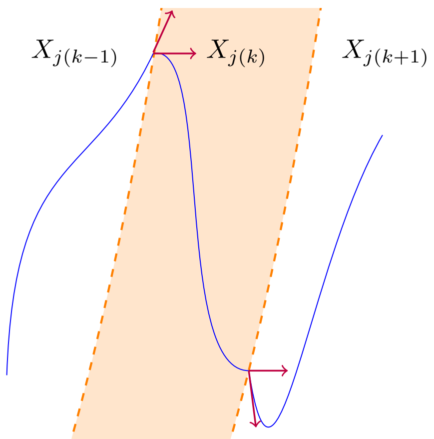

Fig. 1. Illustration of a regular trajectory (in blue) crossing transversally the interfaces between regions.

Let $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ stand for the Hamiltonian function associated with the Mayer optimal control problem (OCP) defined by

$$H(x, u, p) := \langle p, f(x, u) \rangle,$$

for all $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. Our main result is as follows.

Theorem 1. Let (x, u) be a solution to (OCP), which is moreover a regular solution to (CS), and consider the notations introduced in Definitions 1 and 2. Then there exists a piecewise absolutely continuous *costate* $p : [0, T] \to \mathbb{R}^n$, respecting the partition $\mathbb{T} = \{t_k^c\}_{k=0,\dots,N}$ of $[0, T]$, such that:

(1) The adjoint equation $\dot{p}(t) = -\nabla_x f(x(t), u(t))^\top p(t)$ is satisfied for almost every $t \in [0, T]$.
(2) The final condition $p(T) = -\nabla \phi(x(T))$ is satisfied.
(3) At each crossing time t_k^c, the discontinuity condition

$$p^+(t_k^c) - p^-(t_k^c) = \beta_k \nabla F_k(x(t_k^c)),$$

with

$$\beta_k := -\frac{\langle p^+(t_k^c), (f)^+(t_k^c) - (f)^-(t_k^c) \rangle}{\langle \nabla F_k(x(t_k^c)), (f)^-(t_k^c) \rangle},$$

is fulfilled.
(4) For all $k \in \{1, \dots, N\}$ such that $q_{j(k)} = 1$, the Hamiltonian maximization condition

$$u(t) \in \arg\max_{v \in \mathrm{U}} H(x(t), v, p(t)),$$

is fulfilled for almost every $t \in (t_{k-1}^c, t_k^c)$.
(5) For all $k \in \{1, \dots, N\}$ such that $q_{j(k)} = 0$, the averaged Hamiltonian gradient condition

$$\int_{t_{k-1}^c}^{t_k^c} \nabla_u H(x(s), u_k, p(s))\, \mathrm{d}s \in \mathrm{N_U}(u_k).$$

holds true, where $\mathrm{N_U}(u_k)$ stands for the normal cone to U at u_k.

The proof of Theorem 1 is a direct application of the hybrid maximum principle developed in Bayen et al. (2022) that takes into account a regionally switching parameter. Indeed, one has just to see that the control system (CS) can be rewritten as the hybrid control system given by

$$\begin{cases} (x, u) \in \mathrm{AC}([0, T], \mathbb{R}^n) \times \mathrm{L}^\infty([0, T], \mathbb{R}^m), \\ \lambda : [0, T] \to \mathbb{R}^m \text{ is a regionally switching parameter} \\ \qquad\qquad \text{associated with } x, \\ \dot{x}(t) = h(x(t), \lambda(t), u(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_{\text{init}}, \end{cases}$$

where the hybrid dynamics $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by

$$h(x, \lambda, u) := h_j(x, \lambda, u) \text{ if } x \in X_j,$$

where

$$h_j(x, \lambda, u) := \begin{cases} f(x, u) \text{ if } q_j = 1, \\ f(x, \lambda) \text{ if } q_j = 0, \end{cases}$$

for all $(x, \lambda, u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ and all $j \in \mathcal{J}$. Indeed, let us recall that a regionally switching parameter is a function that remains constant while the state position x stays inside a region, and can switch (that is, can change its value) only when the state position x goes from one region to another. To keep this work concise, we do not include the detailed proof of Theorem 1, but we refer to Bayen et al. (2022) for details.

2.3 Comments

• To summarize, Theorem 1 shows that, in each region of type C, the usual Hamiltonian maximization condition holds true, whereas, in each region of type NC, a so-called averaged Hamiltonian gradient condition (in the spirit of the one obtained for optimal sampled-data control problems, see Bourdin and Trélat (2016); Bourdin and Dhar (2019, 2020)) holds true. It is worth mentioning that the latter is implicit in general since u_k intervenes, not only in both sides of the equation, but moreover in the values of x and p along the interval (t_{k-1}^c, t_k^c). Furthermore we do not know in advance the values of t_{k-1}^c and t_k^c. However, as we will see in Section 3, the averaged Hamiltonian gradient condition can be useful to determine the optimal values of the control in regions of type NC.

• In Theorem 1, and as usual in the literature, the Hamiltonian system $(\dot{x}, \dot{p}) = (\nabla_p H, -\nabla_x H)$ is satisfied which implies (together with the other necessary optimality conditions) that the Hamiltonian function

$$t \mapsto H(x(t), u(t), p(t)),$$

is constant almost everywhere over $[0,T]$. Indeed, in a region of type C (with the Hamiltonian maximization condition), one has just to use the standard argumentation (see Pontryagin et al. (1964)). In a region of type NC, the result is straightforward since the control is constant. Finally the discontinuity conditions ensure the constancy at each crossing time t_k^c (see Haberkorn and Trélat (2011)).

• To benefit the most of Theorem 1 (and avoid unnecessary hypotheses), the partition of $\mathbb{R}^n$ must be written so that the number of regions involved is as small as possible. The idea is to avoid, for example, trajectories that would go from a region of type C to another one (which would be redundant from a model point of view).

• In Theorem 1, the discontinuity condition at each crossing time t_k^c is written backward in time. Nonetheless, it can also be written forward in time by noting that

$$\beta_k = -\frac{\langle p^-(t_k^c), (f)^+(t_k^c) - (f)^-(t_k^c)\rangle}{\langle \nabla F_k(x(t_k^c)), (f)^+(t_k^c)\rangle}.$$

• In Definition 2, note that the continuity and limit conditions on the control are superfluous in regions of type NC (since u is constant in such a region).

• Several extensions of Theorem 1 could be of interest and can be easily derived. For the following possible extensions, we refer to Bayen et al. (2022) for details:

- Theorem 1 can be extended to a non-autonomous setting, as well on the dynamics as on the partition of the state space.
- The convexity (resp. closedness) hypothesis on U can be removed by using a generalized version of the normal cone (resp. by assuming that all the limits $u^-(t_k^c)$ and $u^+(t_k^c)$ belong to U).
- One can consider a control constraint set U_j in each region X_j. This would allow to impose the control value in non-control regions. For example, to deal with the case where no control input is allowed in non-control regions, take $\mathrm{U}_j = \{0_{\mathbb{R}^m}\}$ for all $j \in \mathcal{J}$ such that $q_j = 0$.
- One can consider a Bolza cost, involving a Lagrange cost associated with a hybrid Lagrangian function adapted to the partition. This setting would allow to deal with time crisis problems for which the constraint set K is a region of type C (or of type NC).

3. EXAMPLE

In this section, we highlight the use of Theorem 1 on a simple one-dimensional Mayer optimal control problem with one non-control region. Here $n = m = 1$ and $T = 8$.

3.1 Presentation of the example

Consider the partition $\mathbb{R} = \overline{X_1} \cup \overline{X_2} \cup \overline{X_3}$ with

$$\begin{aligned} X_1 &:= \{y \in \mathbb{R} \mid y < -1\}, \\ X_2 &:= \{y \in \mathbb{R} \mid -1 < y < \tfrac{1}{2}\}, \\ X_3 &:= \{y \in \mathbb{R} \mid y > \tfrac{1}{2}\}. \end{aligned}$$

In what follows, we suppose that X_1 and X_3 are of type C (that is $q_1 = q_3 = 1$) and X_2 is of type NC (that is $q_2 = 0$). Now consider the Mayer optimal control problem given by

$$\begin{aligned} &\text{minimize} \quad -x(8), \\ &\text{subject to } (x,u) \in \mathrm{AC}([0,8],\mathbb{R}) \times \mathrm{L}^\infty([0,8],\mathbb{R}), \\ &\qquad \dot{x}(t) = u(t)x(t) + 1 \text{ a.e. } t \in [0,8], \\ &\qquad x(0) = -2, \\ &\qquad u \text{ is constant in the non-control region } X_2, \\ &\qquad u(t) \in [-\tfrac{3}{2}, \tfrac{1}{2}] \text{ a.e. } t \in [0,8]. \end{aligned}$$

The situation is depicted in Figure 2 and the corresponding Hamiltonian is given by

$$H(x,u,p) := p(ux+1),$$

for all $(x,u,p) \in \mathbb{R}^3$.

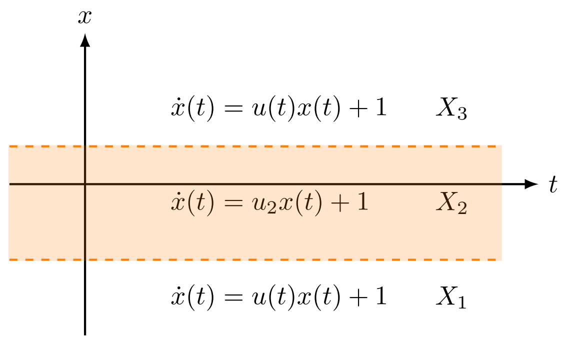

Fig. 2. Illustration of the framework of Section 3.

3.2 Synthesis of an optimal control

In this section, we assume that there exists a solution (x,u), that is regular, and we suppose that it admits exactly two (unknown) crossing times $0 < t_1^c < t_2^c < 8$ and satisfies the following structure:

$$\begin{aligned} t \in [0, t_1^c) &\Rightarrow (x(t), u(t)) \in X_1 \times [-\tfrac{3}{2}, \tfrac{1}{2}], \\ t \in (t_1^c, t_2^c) &\Rightarrow (x(t), u(t)) \in X_2 \times \{u_2\}, \\ t \in (t_2^c, 8] &\Rightarrow (x(t), u(t)) \in X_3 \times [-\tfrac{3}{2}, \tfrac{1}{2}], \end{aligned}$$

where $u_2 \in [-\frac{3}{2}, \frac{1}{2}]$ is unknown and assumed to satisfy $u_2 \neq 0$. Now let us denote by $p : [0,8] \to \mathbb{R}$ the costate provided by Theorem 1. We proceed to the analysis backward in time.

• *Step 1: analysis in the region* X_3. The adjoint equation and final condition give

$$\begin{cases} \dot{p}(t) = -u(t)p(t) \text{ a.e. } t \in [t_2^c, 8], \\ p(8) = 1, \end{cases}$$

and the Hamiltonian maximization condition writes

$$u(t) \in \operatorname*{arg\,max}_{v \in [-\frac{3}{2}, \frac{1}{2}]} x(t)p(t)v \text{ a.e. } t \in (t_2^c, 8).$$

Since $p(t) > 0$ and $x(t) > 0$ over $(t_2^c, 8]$, we get that $u(t) = 1/2$ for almost every $t \in (t_2^c, 8)$ and thus, since $x(t_c^2) = 1/2$, we get that

$$p(t) = e^{4-(t/2)} \quad \text{and} \quad x(t) = \tfrac{5}{2}e^{(t-t_2^c)/2} - 2,$$

for all $t \in (t_2^c, 8]$.

• *Step 2: analysis in the region* X_2. From the discontinuity condition at t_2^c and the adjoint equation, the costate p satisfies

$$\begin{cases} \dot{p}(t) = -u_2 p(t) \text{ a.e. } t \in [t_1^c, t_2^c], \\ p^-(t_2^c) = \frac{5}{2u_2+4} e^{4-(t_2^c/2)}. \end{cases}$$

We get that

$$\begin{aligned} p(t) &= \tfrac{5}{2u_2+4} e^{4-(t_2^c/2)} e^{u_2(t_2^c - t)}, \\ x(t) &= \tfrac{1}{2u_2}((u_2+2)e^{u_2(t-t_2^c)} - 2), \end{aligned} \tag{1}$$

for all $t \in (t_1^c, t_2^c)$. Since $x(t_1^c) = -1$, one deduces that

$$\tfrac{1}{2u_2}((u_2+2)e^{u_2(t_1^c - t_2^c)} - 2) = -1$$

and the relation between t_1^c and t_2^c given by

$$t_2^c = t_1^c + \frac{1}{u_2} \ln\left(\frac{u_2+2}{2(1-u_2)} \right). \tag{2}$$

• *Step 3: analysis in the region* X_1. From the discontinuity condition at t_1^c and the adjoint equation, the costate p satisfies

$$\begin{cases} \dot{p}(t) = -u(t)p(t) \text{ a.e. } t \in [0, t_1^c], \\ p^-(t_1^c) = \frac{5(1-u_2)}{(2u_2+4)(1-u^-(t_1^c))} e^{4-(t_2^c/2)} e^{u_2(t_2^c - t_1^c)}, \end{cases}$$

and the Hamiltonian maximization condition writes

$$u(t) \in \operatorname*{arg\,max}_{v \in [-\frac{3}{2}, \frac{1}{2}]} \ x(t)p(t)v \quad \text{a.e. } t \in (0, t_1^c).$$

Since $p(t) > 0$ and $x(t) < 0$ over $[0, t_1^c]$, we deduce that $u(t) = -3/2$ for almost every $t \in [0, t_1^c]$ and thus, since $x(0) = -2$, we get that

$$\begin{aligned} p(t) &= \tfrac{1}{2} e^{u_2(4-(t_2^c/2))} e^{(3/2)(t - t_1^c)}, \\ x(t) &= \tfrac{2}{3} - \tfrac{8}{3} e^{-(3/2)t}, \end{aligned}$$

for all $t \in [0, t_1^c)$.

• *Step 4: global analysis.* From $x(t_1^c) = -1$, one can easily obtain that $t_1^c = \frac{2}{3}\ln(\frac{8}{5})$. Furthermore, we can now determine the value $u_2 \in [-\frac{3}{2}, \frac{1}{2}]$ thanks to the averaged Hamiltonian gradient condition which writes

$$\gamma(u_2) := \int_{t_1^c}^{t_2^c} x(s)p(s)\, \mathrm{d}s \in \mathrm{N}_{[-\frac{3}{2}, \frac{1}{2}]}(u_2). \tag{3}$$

Using (1) and (2), we find that:

- if $u_2 = 1/2$, then $\mathrm{N}_{[-\frac{3}{2}, \frac{1}{2}]}(u_2) = \mathbb{R}_+$ and $\gamma(u_2) \simeq -26.48 < 0$ which contradicts (3);
- if $u_2 = -3/2$, then $\mathrm{N}_{[-\frac{3}{2}, \frac{1}{2}]}(u_2) = \mathbb{R}_-$ and $\gamma(u_2) \simeq 15.61 > 0$ which contradicts (3).

It follows that $u_2 \in (-\frac{3}{2}, \frac{1}{2})$ and thus (3) implies that $\gamma(u_2) = 0$ which amounts to solving the equation

$$(u_2 - 1)(u_2 + 2) \ln\left(\frac{1 + \frac{u_2}{2}}{1 - u_2} \right) + 3u_2 = 0.$$

This gives us a unique value for $u_2 \in (-\frac{3}{2}, \frac{1}{2})$ given approximately by $u_2 \simeq -0.75$. Finally the optimal control is given by

$$u(t) = \begin{cases} -\frac{3}{2} & \text{a.e. } t \in (0, t_1^c), \\ u_2 & \text{a.e. } t \in (t_1^c, t_2^c), \\ \frac{1}{2} & \text{a.e. } t \in (t_2^c, 8), \end{cases}$$

with $t_1^c = \frac{2}{3}\ln(\frac{8}{5}) \simeq 0.31$ and $t_2^c \simeq 1.68$.

3.3 Comparisons with other control strategies

We end-up this case study with comparisons of the optimal control u obtained in the previous section with different control strategies. To keep this work concise, the computations of this section are omitted.

- First, note that, if the region X_2 was of type C, then the classical PMP would imply that the optimal (permanent) control $\hat{u}$ (associated with the trajectory $\hat{x}$) satisfies
$$\hat{u}(t) = \begin{cases} -3/2 & \text{if } \hat{x}(t) < 0, \\ 1/2 & \text{if } \hat{x}(t) > 0, \end{cases}$$
for almost every $t \in [0, 8]$. However, since X_2 (of type NC) is a strip containing 0, the control $\hat{u}$ is not admissible (since it requires to change its value in the non-control region X_2, see Figure 3).
- Second, from the (nonadmissible) control $\hat{u}$, one might consider the admissible control $u^\perp$ (resp. $u^\dagger$) given by $u^\perp = -3/2$ in both regions X_1 and X_2 (resp. $u^\dagger = -3/2$ in region X_1) and by $u^\perp = 1/2$ in region X_3 (resp. $u^\dagger = 1/2$ in regions X_2 and X_3). The associated trajectory is denoted by $x^\perp$ (resp. $x^\dagger$).

On Figure 3, we depict the trajectories x, $\hat{x}$, $x^\perp$ and $x^\dagger$. As expected, the cost associated with $\hat{x}$ is the best, but is not admissible, while the cost associated with x is admissible and better than the other admissible costs associated with $x^\perp$ and $x^\dagger$. This example shows the relevancy of establishing a PMP in the present context of loss of control since, in general, the optimal constant values in non-control regions do not follow the values of the optimal permanent control obtained with the classical PMP. Furthermore, note that, in contrary to what is usually observed in the classical literature (with permanent controls) when the Hamiltonian is linear with respect to the control, the loss of control can induce optimal constant values in non-control regions that do not saturate the control constraint set U. With this example, we also emphasize that the avegared Hamiltonian gradient condition derived in Theorem 1 allows to determine such optimal values.

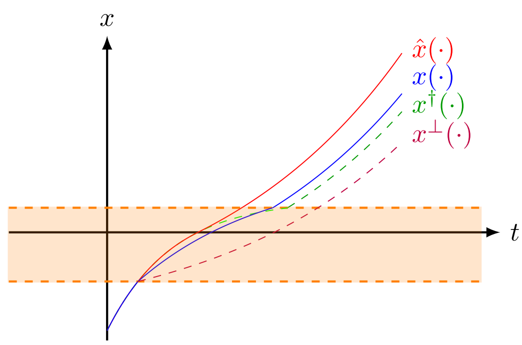

Fig. 3. Trajectories x, $\hat{x}$, $x^\perp$ and $x^\dagger$ from Section 3.3 (zoom on the time interval $[0, 3]$).

4. CONCLUSION AND PERSPECTIVES

In this work, we have introduced a new framework in optimal control theory letting the possibility for a control system to be subject to loss of control depending on its position in a partition of the state space. In our approach, the control value has to be fixed to an admissible value as long as the system belongs to a non-control region but we do not know in advance how long the system stays in such a region. The corresponding optimal constant value satisfies (and possibly is determined by) the averaged Hamiltonian gradient condition. We believe that this setting differs from other frameworks covered by hybrid optimal control problems or state constrained optimal control problems (see, *e.g.*, Frankowska and Osmolovskii (2018)) and that it could be employed in various practical situations such as in aerospace (in particular for the determination of an optimal control strategy when a spacecraft enters into a shadow zone). Future works could focus on the determination of optimal control policies in this framework for SIR models or in population models in the context of time crisis problems when one is unable to control in the non-constraint set (see, *e.g.*, Bayen and Rapaport (2019)). We are also interested in extending the necessary optimality conditions obtained in this paper to the case of feedback controls in non-control regions (instead of frozen controls) and to the context of final state constraints, and in developing Riccati theory for linear control systems subject to loss of control. Also note that, in this paper, we did not discuss the existence of a solution to (OCP) which may be a difficult question due to the presence of non-control regions. So, in Theorem 1, we have assumed that there exists a solution to (OCP), moreover with a finite number of crossing times, excluding that way other possible solutions with more complicated structures such as chattering, boundary arcs, tangential crossing, etc. On the other hand, note that considering non-control regions may impact the controllability of (CS) but we did not discuss controllability issues here. All these subjects constitute interesting perspectives for further research works.

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