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ORBITAL EQUIVALENCE CLASSES OF FINITE COVERINGS OF GEODESIC FLOWS

THIERRY BARBOT AND SÉRGIO R. FENLEY

Abstract − Let M be a closed 3-manifold admitting a finite cover of degree n along the fibers over the unit tangent bundle of a connected closed surface. We prove that if n is odd, there is only one Anosov flow on M up to orbital equivalence, and if n is even, there are exactly two orbital equivalence classes of Anosov flows on M.

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1. Introduction

Let Σ be a closed connected orientable surface of genus $g > 1$, equipped with a negatively curved Riemannian metric, and let $T^1\Sigma$ be the unit tangent bundle. The geodesic flow Φ_0^t on $T^1\Sigma$ is a typical *Anosov flow* (see Section 3.1 for a definition of the geodesic flow, and Definition 2.1 for a definition of Anosov flows).

E. Ghys, in [Gh1], proved that, up to finite covers, geodesic flows are essentially the only possible Anosov flows on circle bundles over a closed surface. More precisely, if Φ^t is an Anosov flow on a closed manifold M admitting a circle fibration over the closed surface Σ, then there is a finite covering $q : M \to T^1\Sigma$ such that Φ^t is a reparametrization of the lifting in M by q of the geodesic flow on $T^1\Sigma$ (see Theorem 3.3).

Observe that the finite covering q is *along the fibers* (see Section 3.3) − roughly this means that both fibrations have the same base space Σ. This implies that if M is homeomorphic the unit tangent bundle $T^1\Sigma$ itself, then q is a homeomorphism, and that, in this case, Φ^t is merely orbitally equivalent to the geodesic flow itself (see Definition 2.3).

This is not true in the general case. More precisely: let M be a closed 3-manifold admitting a circle fibration over Σ. Ghys' Theorem implies that in order to support some Anosov flow, M must be a finite cover of $T^1\Sigma$. The degree n of this finite cover is a topological invariant of M: any other finite covering $q : M \to T^1\Sigma$ has degree n (see beginning of Section 3.3). For a fixed interger n, there are infinitely

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many such finite coverings, providing different Anosov flows on M. Hence a natural question is to ask if different finite coverings can provide different orbital equivalence classes of Anosov flows? The answer is given by Theorem 4.5, that we restate here:

Main Theorem : *Let $q_0 : M \to T^1(\Sigma)$ be a finite covering of degree n along the fibers. Then, if n is odd, there is only one Anosov flow on M up to orbital equivalence. If n is even, there are exactly two orbital equivalence classes of Anosov flows.*

The proof of the Main Theorem has a dynamical part which is mainly contained in Ghys' Theorem. After that the proof is reduced in this article to a study of the action of the (extended) mapping class group $\mathrm{Mod}^{\pm}(\Sigma)$ on the set $\mathfrak{G}_n$ of subgroups of $\widetilde{\Gamma} = \pi_1(T^1\Sigma)$ corresponding to finite coverings of degree n along the fibers. More precisely, there is an exact sequence

$$0 \to \mathbb{Z} \to \widetilde{\Gamma} \to \pi_1(\Sigma) \to 1$$

and elements of $\mathfrak{G}_n$ are subgroups of $\widetilde{\Gamma}$ of index n such that the projection on $\pi_1(\Sigma)$ is surjective (see Definition 3.16).

It is known (and recalled in Section 3.4) that the mapping class group $\mathrm{Mod}^{\pm}(\Sigma)$ acts naturally on $\mathfrak{G}_n$. In Proposition 3.17 we prove that there is a one-to-one correspondence between orbital equivalence classes of Anosov flows in M and $\mathrm{Mod}^{\pm}(\Sigma)$-orbits in $\mathfrak{G}_n$.

Hence, the core of the proof is Proposition 4.4, where we compute explicitly how the Lickorish generators of $\mathrm{Mod}(\Sigma)$ act on $\mathfrak{G}_n$, and which allows us (with a little more work) to compute the number of $\mathrm{Mod}^{\pm}(\Sigma)$-orbits in $\mathfrak{G}_n$ according to the parity of n.

Our Main Theorem is closely related to Giroux's classification of universally tight contact structures, with wrapping number $-n$, on a circle bundle M ([Gi]). See Remark 4.9 for more details.

In addition, in Section 3.3 we describe the space of isotopy classes of Anosov flows on a given circle bundle over a closed surface (Corollary 3.15). There are infinitely many, but there is only one up to vertical Dehn twists as defined in Remark 3.7.

Finally we note the recent work of Barthelme, Mann and collaborators on classifying transitive Anosov (and pseudo-Anosov) flows up to orbital equivalence using the knowledge of the spectrum [Ba-Ma, BFeM, BFrM]. The *spectrum* is the set of free homotopy classes of curves in M which are represented by periodic orbits of the flow. In particular perhaps it might be possible to prove the orbital equivalence result of Subsection 3.3 (Proposition 3.17) using the results of Barthelme, Mann and collaborators.

2. Background

2.1. Anosov flows – definitions.

Definition 2.1. *(Anosov flow) Let Φ^t be a non-singular C^k-flow ($k \geq 1$) on a closed connected 3-manifold M, equipped with an auxiliary Riemannian metric. We say that Φ^t is Anosov if the tangent bundle of M admits a continuous $d\Phi^t$ - invariant splitting:*

$$TM = \mathbb{R}\Phi \oplus E^{ss} \oplus E^{uu}$$

such that:

- $\mathbb{R}\Phi$ is the one-dimensional bundle tangent to the orbits of the flow,

- There are two positive real numbers a, C, such that, for every vector v^s in E^{ss} (respectively v^u in E^{uu}) and for every $t > 0$, the following inequalities hold:

$$\begin{aligned} \|d\Phi^t(v^s)\| &\leq Ce^{-at}\|v^s\| \\ \|d\Phi^t(v^u)\| &\geq \frac{1}{C}e^{at}\|v^u\| \end{aligned}$$

The invariant bundles E^{ss}, E^{uu} are called the *strong stable and strong unstable bundles*, respectively. They are usually only Hölder continuous, nevertheless they are uniquely integrable, tangent to one-dimensional foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, called the *strong stable and unstable foliations* [An, Ha-Ka].

The *weak stable and unstable bundles* $E^s := \mathbb{R}\Phi \oplus E^{ss}$ and $E^u := \mathbb{R}\Phi \oplus E^{uu}$ are uniquely integrable too, the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ they integrate to are the *weak stable and unstable foliations.* These foliations

are C^1. In fact these foliations are even $C^{1+\text{Zygmund}}$ in the case that the flow preserves some volume form on M and the flow is C^k with $k \geq 3$ (Theorem 3.1 of [Hu-Ka]).

Sometimes we will assume that M is orientable and/or the foliations $\mathcal{F}^s$, $\mathcal{F}^u$ are transversely orientable. This can always be achieved in a cover of order at most 4.

The intersection foliation $\Phi = \mathcal{F}^s \cap \mathcal{F}^u$ is an oriented 1-dimensional foliation, whose leaves are the orbits of Φ^t, oriented by the time direction. The weak foliations $\mathcal{F}^s$, $\mathcal{F}^u$ only depend on Φ and not on the parametrization of the flow, contrary to the strong foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$.

In this article we will use the following terminology, which may be different from other articles:

Definition 2.2. *An* Anosov foliation *is a foliation $\Phi = \mathcal{F}^s \cap \mathcal{F}^u$, oriented by the time direction, where $\mathcal{F}^s$, $\mathcal{F}^u$ are the weak stable and unstable foliations of some Anosov flow.*

We also remind the following definition, for the reader's convenience:

Definition 2.3. *A C^r-*orbital equivalence *between two flows (M, Φ^t) and (N, Ψ^s) is a C^r-diffeomorphism $f : M \to N$ mapping oriented orbits onto oriented orbits: there exists a continuous map $u : \mathbb{R} \times M \to \mathbb{R}$, increasing with the first factor, and such that, for every (t, x) in $\mathbb{R} \times M$ we have:*

$$f(\Phi^t(x)) = \Psi^{u(t,x)}(f(x)).$$

*The flows are then C^r-*orbitally equivalent*. If moreover the map u satisfies $u(t, x) = t$, then the orbital equivalence is a (C^r-)*conjugation*. Here r could be 0, in which case f is only a homeomorphism.*

Observe that the orbital equivalence maps the foliation Φ onto the foliation Ψ. Orbital equivalences between Anosov flows can therefore be defined as conjugacies between Anosov foliations preserving the orientation. In this article we will consider orbital equivalences between Anosov flows and therefore omit the reference to Anosov flows. Orbital equivalences map weak leaves onto weak leaves.

From now on, Φ^t denotes a C^k-Anosov flow ($k \geq 1$) on a closed 3-manifold M.

2.2. **Orbit space and leaf spaces of Anosov flows.** We denote by $\pi : \widetilde{M} \to M$ the universal covering of M, and by $\pi_1(M)$ the fundamental group of M, considered as the group of deck transformations on $\widetilde{M}$. The foliations lifted to $\widetilde{M}$ are denoted by $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$. If $x \in M$ let $\mathcal{F}^s(x)$ denote the leaf of $\mathcal{F}^s$ containing x. Similarly one defines $\mathcal{F}^u(x)$ and in the universal cover $\widetilde{\mathcal{F}}^s(x), \widetilde{\mathcal{F}}^u(x)$. If θ is a leaf of Φ, we denote by $\mathcal{F}^s(\theta)$, $\mathcal{F}^u(\theta)$ the weak leaves containing θ. Let also $\widetilde{\Phi}^t$ be the lifted flow to $\widetilde{M}$. We adopt similar conventions denoting by $\widetilde{\mathcal{F}}^s(\tilde{\theta})$, $\widetilde{\mathcal{F}}^u(\tilde{\theta})$ the weak stable and unstable leaves of $\tilde{\theta}$.

We denote by $\mathcal{O}$ the orbit space $\widetilde{M}/\widetilde{\Phi}$ and by $\Theta : \widetilde{M} \to \mathcal{O}$ the quotient map. It is remarkable that $\widetilde{M}$ is always homeomorphic to $\mathbb{R}^3$, $\mathcal{O}$ diffeomorphic to $\mathbb{R}^2$, and that Θ is a $\pi_1(M)$-equivariant $\mathbb{R}$-principal fibration ([Ba1], [Fe-Mo]). We denote by $\mathcal{O}^s$, $\mathcal{O}^u$ the 1-dimensional foliations induced by $\widetilde{\mathcal{F}}^s$, $\widetilde{\mathcal{F}}^u$ on $\mathcal{O}$. The leaves of these foliations are properly embedded lines.

We denote the leaf spaces $\widetilde{M}/\widetilde{\mathcal{F}}^s$, $\widetilde{M}/\widetilde{\mathcal{F}}^u$ by respectively Q^s, Q^u. Observe that they are canonically identified with the leaf spaces $\mathcal{O}/\mathcal{O}^s$, $\mathcal{O}/\mathcal{O}^u$. A particular case of interest for us is the case of $\mathbb{R}$-*covered Anosov flows,* i.e. the case where Q^s or Q^u is homeomorphic to the real line. In fact if one of them is homeomorphic to the real line, then both of them are [Ba1]. We fix homeomorphisms of these with $\mathbb{R}$. This induces total order in Q^s, Q^u (both homeomorphic to $\mathbb{R}$). If $\mathcal{F}^s, \mathcal{F}^u$ are transversely oriented this total order is preserved by deck transformations.

In addition the following is true:

Theorem 2.4. *(*[Ba2, Théorème 2.10]*) Let (M, Φ^t) be an $\mathbb{R}$-covered Anosov flow. Assume that (M, Φ^t) is not orbitally equivalent to the suspension of an Anosov diffeomorphism of the torus. Then, the map $\Upsilon : \mathcal{O} \to Q^s \times Q^u$ sending an orbit $\tilde{\theta}$ to the pair $(\widetilde{\mathcal{F}}^s(\tilde{\theta}), \widetilde{\mathcal{F}}^u(\tilde{\theta}))$ is an homeomorphism onto its image, which is:*

$$\Omega := \{(x, y) \in Q^s \times Q^u \mid \alpha(x) < y < \beta(x)\}$$

where α and β are two homeomorphisms from Q^s into Q^u satisfying, for every γ in $\pi_1(M)$:

$$\alpha \circ \gamma = \gamma \circ \alpha \text{ and } \beta \circ \gamma = \gamma \circ \beta \text{ if } \gamma \text{ preserves the orientation of } Q^s \text{ and } Q^u$$

$$\alpha \circ \gamma = \gamma \circ \beta \text{ and } \beta \circ \gamma = \gamma \circ \alpha \text{ if } \gamma \text{ reverses the orientation of } Q^s \text{ and } Q^u$$

Remark 2.5. The $\mathbb{R}$-covered Anosov flows in Theorem 2.4 are called *skewed* Anosov flow. We refer the reader to [Ba1, Fen] for a more detailed study of these flows and their properties. For example, it follows from these papers that for every γ in $\pi_1(M)$, γ preserves the orientation of Q^s if anf only if it preserves the orientation of Q^u, so that the two cases considered at the end of Theorem 2.4 indeed cover all the possible cases.

Here we first stress that α, β are homeomorphims, and not just continuous maps, which is a big part of the content of the theorem. In particular it follows from Theorem 2.4 that the composition $\tau_s = \alpha^{-1} \circ \beta$ is a homeomorphism without fixed points of Q^s into itself satisfying for every γ in $\pi_1(M)$:

$$\tau_s \circ \gamma = \gamma \circ \tau_s \text{ if } \gamma \text{ preserves the orientation of } Q^s \text{ and } Q^u$$

$$\tau_s \circ \gamma = \gamma \circ \tau_s^{-1} \text{ if } \gamma \text{ reverses the orientation of } Q^s \text{ and } Q^u$$

We can furthermore define a map $\zeta : \mathcal{O} \to \mathcal{O}$ which is conjugated by Υ to the map:

$$(x, y) \mapsto (\alpha^{-1}(y), \beta(x)).$$

This map ζ satisfies:

$$\zeta \circ \gamma = \gamma \circ \zeta \text{ if } \gamma \text{ preserves the orientation of } Q^s \text{ and } Q^u$$

$$\zeta \circ \gamma = \gamma \circ \zeta^{-1} \text{ if } \gamma \text{ reverses the orientation of } Q^s \text{ and } Q^u$$

Observe that the map ζ permutes the foliations $\mathcal{O}^s$ and $\mathcal{O}^u$. By an abuse of notation we will identity $\mathcal{O}$ and Ω by the homeomorphism Υ. This induces an action of ζ on Ω.

Remark 2.6. The homeomorphisms ζ, τ_s, α and β are Hölder continuous (see [Ba3, Proposition 2.1]).

Remark 2.7. The image under Υ of a stable leaf $\mathcal{O}^s(\tilde{\theta})$ is a vertical segment contained in Ω that we will call later a *stable leaf* in Ω. Similarly, images under Υ of unstable leaves $\mathcal{O}^u(\tilde{\theta})$ will be horizontal segments that we will call later *unstable leaves* in Ω.

Corollary 2.8. *Let (M, Φ^t) be a skewed $\mathbb{R}$-covered Anosov flow. Then, the only homeomorphisms from $\mathcal{O}$ into itself commuting with the action of $\pi_1(M)$ are the powers ζ^k ($k \in \mathbb{Z}$) of ζ. If the homeomorphism is induced by a self orbit equivalence of Φ, then it is of the form ζ^{2k}, where $k \in \mathbb{Z}$.*

Proof. Let us consider some homeomorphism $f : \mathcal{O} \to \mathcal{O}$ commuting with the action of $\pi_1(M)$. We will abuse notation and also denote by f the action induced on Ω by its conjugate under Υ.

Let γ be a non-trivial element of $\pi_1(M)$ fixing a point (x, y) of Ω. Then, the other fixed points of γ are the elements of the orbits of (x, y) under ζ. Therefore, there is some integer k such that:

$$f(x, y) = \zeta^k(x, y)$$

Hence, (x, y) is a fixed point of $\zeta^{-k} \circ f$.

Let (x_1, y_1) fixed by some γ_1 non trivial and (x_1, y_1) sufficiently close to (x, y), so that (x_1, y) and (x, y_1) are in Ω, and in addition $\zeta^{-k} \circ f(x_1, y_1)$ is very close to (x_1, y_1) also. The previous paragraph implies that there is j so that $\zeta^{-j} \circ f(x_1, y_1) = (x_1, y_1)$. Hence the stable leaf of $\zeta^{-k}(x_1, y_1)$ intersects the unstable leaf of $\zeta^{-j}(x_1, y_1)$ and vice versa. This immediately implies that $j = k$. In particular the arbitrarily near is uniform, that is there is $\epsilon > 0$ so that if the orbit of (x, y) and the orbit of (x_1, y_1) are ϵ close at some point then the above happens.

Since Φ is $\mathbb{R}$-covered, it is topologically transitive [Ba1], and therefore the union of periodic orbits are dense in M. It means that fixed points of non trivial elements of $\pi_1(M)$ are dense in Ω. It follows that elements of Ω with non-trivial $\pi_1(M)$-stabilizers can be connected by a finite sequence of elements with non-trivial $\pi_1(M)$-stabilizers such that two successive elements of the sequence are sufficiently close one to the other in the preceding meaning.

As a conclusion, the integer k at elements of Ω with non-trivial $\pi_1(M)$-stabilizer is the same at each of them. In other words, there is an integer k such that $\zeta^{-k} \circ f$ preserves every element of Ω with non-trivial $\pi_1(M)$-stabilizer. Since the union of these elements is dense in Ω, the equality $f = \zeta^k$ holds everywhere. This proves the first statement of the corollary.

To prove the second statement notice that if the homeomorphism f comes from an orbit equivalence, then it sends the weak stable foliation to itself and likewise for the weak unstable foliation. By Remark 2.5, it now follows that f must be an even power of ζ. The corollary follows. □

We remark that more generally one can consider orientation reversing orbit equivalences as well: they are "weak" orbit equivalences which reverse the direction of the flow.

2.3. **Recovering the Anosov flow from the orbit space.** In this Section, we see how, from the data of the orbit space $\mathcal{O}$ and the $\pi_1(M)$-action on it, to recover the Anosov flow itself up to orbital equivalence. The flow (M, Φ^t) is not assumed to be $\mathbb{R}$-covered, nor topologically transitive.

Theorem 2.9. *(*[Ba1, Théorème 3.4]*) Let (M, Φ^t) and (N, Ψ^s) be two C^k-Anosov flows on closed connected 3-manifolds ($k \geq 1$). Assume that there exists an isomorphism $\rho : \pi_1(M) \to \pi_1(N)$ and an equivariant C^r-diffeomorphism $\bar{f} : \mathcal{O} \to Q^\Psi$ between the orbit spaces ($1 \leq r \leq k$). Then, $\bar{f}$ lifts to an equivariant C^r-diffeomorphism between the universal coverings. In other words, either (M, Φ^t) is orbitally equivalent to (N, Ψ^s), or (M, Φ^t) is orbitally equivalent to (N, Ψ^{-s}) (i.e. the second flow with the time direction reversed). If moreover $\bar{f}$ maps the stable foliation $\mathcal{O}^s(\Phi)$ onto the stable foliation $\mathcal{O}^s(\Psi)$, then the first case occurs, i.e. (M, Φ^t) is orbitally equivalent to (N, Ψ^s).*

It follows in particular that, according to Remark 2.5, an $\mathbb{R}$-covered Anosov flow admitting no global cross-section is weak orbitally equivalent to its own inverse: the map ζ defined in Remark 2.5 lifts to some weak orbital equivalence $F_0 : M \to M$ between Φ and its inverse. Moreover, F_0 is isotopic to the identity.

Observe also the following consequence of Corollary 2.8:

Corollary 2.10. *Let (M, Φ^t) be a skewed $\mathbb{R}$-covered Anosov flow. Let $F : M \to M$ be an orbital equivalence between Φ and itself. Assume that F is isotopic to the identity. Then, F is isotopic along Φ to some power F_0^{2k} of F_0.*

This follows because the orbital equivalence has a lift to $\widetilde{M}$ which commutes with all deck translations.

3. Classification of finite coverings of geodesic flows up to isotopy and orbital equivalence

3.1. **Geodesic flows.** In this Section, Σ is a closed connected oriented surface of genus $g > 1$, we denote by $\bar{\Gamma}$ its fundamental group, and by $p^0 : M_1(\Sigma) \to \Sigma$ the *positive* projective tangent bundle of Σ, i.e. the quotient of the tangent bundle with the zero section removed by the relation identifying two vectors if they are proportional up to a *positive* real number. This definition avoids the choice of a peculiar Riemannian metric, but $M_1(\Sigma)$ clearly identifies with the unit tangent bundle for any Riemannian metric on Σ.

According to [An], the geodesic flow of any negatively curved metric on Σ is an Anosov flow on $M_1(\Sigma)$. Moreover, since the Teichmüller space Teich(Σ) is connected, and since the space of negatively curved metrics in a given conformal class is connected, any pair of negatively curved metrics on Σ can be joined by a path of negatively curved metrics. It follows from structural stability of Anosov flows that Anosov geodesic flows on $M_1(\Sigma)$ are isotopic one to the other. Hence, we can speak of *the Anosov geodesic flow* of Σ.

Let us review alternative ways to define geodesic flows for hyperbolic metrics, each of them being useful in the rest of the paper:

3.1.1. *Geodesic flows as algebraic flows.* Let $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ be the universal covering of $\mathrm{PSL}(2, \mathbb{R})$, and let $\widetilde{\Gamma}$ be the full preimage in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of the uniform lattice $\bar{\Gamma}$. Then, $M_1(\Sigma)$ is diffeomorphic to the quotient $\widetilde{\Gamma} \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R})$, and the geodesic flow is the flow induced by the action on the right of the 1-dimensional Lie subgroup whose projection in $\mathrm{PSL}(2, \mathbb{R})$ is the subgroup D represented by diagonal matrices of the form:

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

More generally, every finite covering of the geodesic flow has a similar description, where $\widetilde{\Gamma}$ is replaced by some finite index subgroup of itself.

3.1.2. *Geodesic flow on the space of triples.* This construction is described by M. Gromov in [Gro] and therein attributed to M. Morse. Realize $\bar{\Gamma}$ as a torsion-free uniform lattice in $\mathrm{PSL}(2,\mathbb{R})$, hence as a discrete group of projective transformations of the (oriented) circle $\mathbb{R}P^1$. For now and future use let

$$X \;=\; \{\; (x,y,z) \in (\mathbb{R}P^1)^3,\; x \neq y \neq z \neq x,\; x < y < z \;\text{ for the cyclic order of } \mathbb{R}P^1 \;\}$$

For every (x,y,z) in X let p be the unique point in the hyperbolic plane $\mathbb{H}^2$ lying in the geodesic (xy) of extremities x and y such that the geodesic ray starting from p and going to z is orthogonal to (xy), and let v be the unit vector tangent to (xy) at p pointing towards x. The map $(x,y,z) \mapsto (p,v)$ identifies X with the unit tangent bundle of the hyperbolic plane $T^1\mathbb{H}^2$. It follows that the diagonal action of $\pi_1(M)$ on X is free and properly discontinuous, and that the quotient space is homeomorphic to $M_1(\Sigma)$. Moreover, the geodesic flow on $M_1(\Sigma)$ corresponds to the flow on X preserving x and y and moving z from y to x. Another choice of realization of $\bar{\Gamma}$ as an uniform lattice in $\mathrm{PSL}(2,\mathbb{R})$ leads to the same action on $\mathbb{R}P^1$ up to topological conjugacy, hence to the same flow up to orbital equivalence.

3.1.3. *Geodesic flows on the projectived tangent bundle over the band.* Let us consider once more the universal covering $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ and the uniform lattice $\widetilde{\Gamma}$. Observe that $\widetilde{\Gamma}$ acts naturally on the (oriented) universal covering $\widetilde{\mathbb{R}P^1}$ of $\mathbb{R}P^1$. The center of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ is the group of deck transformations of the universal covering $\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$; it is a cyclic group generated by an increasing map $\delta : \widetilde{\mathbb{R}P^1} \to \widetilde{\mathbb{R}P^1}$.

The identification between X (of the previous subsection) and $T^1\mathbb{H}^2$ makes clear that the lifting in X of the weak stable leaves are the level sets of the projection map $(x,y,z) \mapsto x$. Therefore, the leaf space of this weak stable foliation is $\mathbb{R}P^1$. Hence, the leaf spaces Q_0^s and Q_0^u for the geodesic flow are both $\widetilde{\Gamma}$-equivariantly isomorphic to $\widetilde{\mathbb{R}P^1}$, and the image of the map $\upsilon : \mathcal{O}_0 \to Q_0^s \times Q_0^u$ appearing in Theorem 2.4 is $\widetilde{\Gamma}$-equivariantly isomorphic to the region in $\widetilde{\mathbb{R}P^1} \times \widetilde{\mathbb{R}P^1}$ between the graphs of the identity map of $\widetilde{\mathbb{R}P^1}$ and the map δ.

Remark 3.1. Let $p_*^0 : \widetilde{\Gamma} \to \bar{\Gamma}$ be the projection map. Let $\bar{\gamma}$ be a non-trivial element of $\bar{\Gamma}$. It fixes two points in $\mathbb{R}P^1$, hence it admits a lift $\widetilde{\gamma}$ in $\widetilde{\Gamma}$ that admits fixed points in $\widetilde{\mathbb{R}P^1}$. More precisely, $\bar{\gamma}$ admits an attracting fixed point, and a repelling fixed point. Therefore, $\widetilde{\gamma}$ admits a δ-orbit of attracting fixed points, and a δ-orbit of repelling fixed points. Other elements of $(p_*^0)^{-1}(\bar{\gamma})$ are elements of $\widetilde{\Gamma}$ of the form $\widetilde{\gamma}\delta^k$ for some integer k. It follows that $\widetilde{\gamma}$ is the unique element $(p_*^0)^{-1}(\bar{\gamma})$ admitting fixed points in $\widetilde{\mathbb{R}P^1}$.

We thus have defined a canonical section $\sigma_0 : \bar{\Gamma} \to \widetilde{\Gamma}$, the one associating to every element $\bar{\gamma}$ the unique element in its preimage by p_*^0 fixing at least one point in $\widetilde{\mathbb{R}P^1}$. This map is not a group homomorphism. Actually, it defines a cocycle $c : \bar{\Gamma} \times \bar{\Gamma} \to \mathbb{Z}$, where $c(\bar{\gamma}_1, \bar{\gamma}_2)$ is the unique integer k satisfying:

$$\sigma_0(\bar{\gamma}_1\bar{\gamma}_2) = \delta^k \sigma_0(\bar{\gamma}_1)\sigma_0(\bar{\gamma}_2)$$

This cocycle represents a cohomology class in $\mathrm{H}^2(\bar{\Gamma},\mathbb{Z})$ which is the Euler class. This cocycle is not trivial, meaning it is not a coboundary. As we will also see later, this will not be the case for its projection in $\mathbb{Z}/n\mathbb{Z}$, when n divides $2g-2$. Actually, as proved in Lemma 3.2, this cocycle takes value in $\{-1,0,+1\}$, meaning that it represents a *bounded* cohomology class, which is the bounded Euler class (see [Gh2]).

The following lemma will be extremely useful:

Lemma 3.2. *Choose an arbitrary hyperbolic metric on Σ, so that $\bar{\Gamma}$ is realized as a uniform lattice in $PSL(2,\mathbb{R})$. Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ be two elements of $\bar{\Gamma}$, and let Δ_1, Δ_2 be their oriented axis in $\mathbb{H}^2$ as hyperbolic isometries of the hyperbolic plane. If Δ_1 and Δ_2 intersect transversely, or if they do not intersect but have the same direction, then $c(\bar{\gamma}_1,\bar{\gamma}_2) = 0$.*

Proof. By "same direction" we mean that the attracting fixed points of $\bar{\gamma_1}, \bar{\gamma_2}$ do not separate in $\mathbb{R}P^1$ the repelling fixed points of these elements.

Since $\bar{\Gamma}$ is realized as a uniform lattice in $\mathrm{PSL}(2,\mathbb{R})$, the group $\widetilde{\Gamma}$ is realized as a lattice in $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$.

Let x_1, x_2 be the attracting fixed point in $\mathbb{R}P^1$ of $\bar{\gamma}_1$,$\bar{\gamma}_2$, respectively, and let y_1, y_2 be their repelling fixed points.

Let now $\gamma_i = \sigma_0(\bar{\gamma}_i)$. Let us fix one attracting fixed point $\tilde{x}_1$ for γ_1 in $\widetilde{\mathbb{R}P^1}$. Then the interval $]\tilde{x}_1, \delta\tilde{x}_1[$ contains one and only one repelling fixed point $\tilde{y}_1$ for γ_1, and only two fixed points $\tilde{x}_2$, $\tilde{y}_2$ of γ_2, respectively attracting and repelling.

We have, for the cyclic order on $\mathbb{R}P^1$, only four possibilities:

(1) $x_1 < x_2 < y_1 < y_2$, or
(2) $x_1 < y_2 < y_1 < x_2$, or
(3) $x_1 < x_2 < y_2 < y_1$, or
(4) $x_1 < y_1 < y_2 < x_2$.

In all these cases, there is a closed interval in $\mathbb{R}P^1$ whose endpoints are x_1 and x_2, and whose interior does not contain y_1 and y_2. This interval lifts to a closed interval I_0 in $\widetilde{\mathbb{R}P^1}$ bounded by two attracting fixed points of γ_1, γ_2, and containing no other fixed point of γ_1, γ_2. It follows that both γ_1 and γ_2 send the interval I_0 into itself. It follows that:

$$\gamma_1\gamma_2(I_0) \subset I_0$$

.

Therefore, $\gamma_1\gamma_2$ admits a fixed point in I_0, and $\gamma_1\gamma_2$ is $\sigma_0(\bar{\gamma}_1\bar{\gamma}_2)$. The Lemma is proved. □

3.2. **Anosov flows on the unit tangent bundle over a closed surface.** A fundamental theorem is that the geodesic flow is essentially the unique Anosov flow on circle bundles - and our purpose here is to discuss what exactly the term "essentially" means.

Theorem 3.3 ([Gh1]). *Let Φ^t be an Anosov flow on a closed orientable manifold M admitting a circle fibration over a closed connected oriented surface Σ of genus $g > 1$. Then, there is a finite covering $p : M \to M_1(\Sigma)$ such that Φ^t is a reparametrization of the lifting in M by p of the Anosov geodesic flow of Σ.*

In the rest of this Section, we study what are the isotopy classes of Anosov flows on the unit tangent bundle $M_1(\Sigma)$ itself. Let us introduce some notations:

– Equip $p^0 : M_1(\Sigma) \to \Sigma$ with a structure of principal $\mathbb{S}^1$-bundle (for example, select a metric on Σ and then have the action to be rotation by an angle in $\mathbb{S}^1$ − this uses that Σ is oriented).

– Let $\mathfrak{I}_1$ be the set of isotopy classes of Anosov flows on $M_1(\Sigma)$.

– The fundamental group of Σ has a presentation:

$$\bar{\Gamma} = \langle \bar{a}_i, \bar{b}_i \ (i = 1, ..., g) \mid [\bar{a}_1, \bar{b}_1][\bar{a}_2, \bar{b}_2]...[\bar{a}_g, \bar{b}_g] = 1 \rangle \tag{1}$$

– The map $p^0 : M_1(\Sigma) \to \Sigma$ is a fibration by circles of Euler class $2g-2$. In particular, the fundamental group $\widetilde{\Gamma}$ of $M_1(\Sigma)$ has a presentation:

$$\widetilde{\Gamma} = \langle h, a_i, b_i \ (i = 1, ..., g) \mid [a_1, b_1][a_2, b_2]...[a_g, b_g] = h^{2g-2} \rangle \tag{2}$$

where the projections of the a_i's and b_i's are generators $\bar{a}_i$ and $\bar{b}_i$ of $\bar{\Gamma}$, and h is represented by the oriented fibers of p.

– Let $p^0_* : \widetilde{\Gamma} \to \bar{\Gamma}$ be the map induced as the fundamental group level by $p^0 : M_1(\Sigma) \to \Sigma$.

– Let $\mathrm{Mod}(\Sigma)$ be the mapping class group of Σ, i.e. the group of orientation-preserving homeomorphisms of Σ up to isotopy.

– Let $\mathrm{Mod}^{\pm}(\Sigma)$ be the extended mapping class group of Σ, i.e. the group of homeomorphisms of Σ up to isotopy.

– Let $\mathrm{Mod}(M_1(\Sigma))$ be the extended mapping class group of $M_1(\Sigma)$, i.e. the group of homeomorphisms of $M_1(\Sigma)$ up to isotopy. We will see in Remark 3.9 that homeomorphisms of $M_1(\Sigma)$ are all orientation-preserving.

– Note that $\ker p^0_*$ is generated by h, and of course we require $\bar{a}_i = p^0_*(a_i)$ and $\bar{b}_i = p^0_*(b_i)$.

Remark 3.4. All the manifolds we consider are surfaces or irreducible Haken 3-manifolds. Therefore, homeomorphisms are isotopic if and only if they are homotopically equivalent ([Wald]).

Remark 3.5. In this context, the homeomorphism of $\mathcal{O}$ induced by h is ζ^2, where ζ is the homeomorphism defined in Remark 2.5. Therefore, in this context, Corollary 2.10 becomes:

Corollary 3.6. *Let $F : M_1(\Sigma) \to M_1(\Sigma)$ be an orbital equivalence between the geodesic flow Φ_0 and itself. Assume that F is isotopic to the identity. Then, F is an isotopy along the orbits of Φ_0.*

This is because ζ^2 sends any orbit to a deck translate of it.

According to Baer-Dehn-Nielsen Theorem [Fa-Ma], $\mathrm{Mod}^{\pm}(\Sigma)$ is isomorphic to $\mathrm{Out}(\bar\Gamma)$, i.e. the quotient of the group $\mathrm{Aut}(\bar\Gamma)$ of automorphisms of $\bar\Gamma$ by the normal subgroup comprising inner automorphisms. The mapping class group $\mathrm{Mod}(\Sigma)$ is isomorphic to $\mathrm{Out}^+(\bar\Gamma)$, i.e. the quotient by inner automorphisms of the group of automorphism of $\bar\Gamma$ preserving the fundamental class, see Theorem 8.1 of [Fa-Ma].

Since Σ has higher genus, every homeomorphism of $M_1(\Sigma)$ is isotopic to a homeomorphism preserving the fibers of $p^0 : M_1(\Sigma) \to \Sigma$. Therefore, there is a well-defined exact sequence:

$$1 \to K_1 \to \mathrm{Mod}(M_1(\Sigma)) \to \mathrm{Mod}^{\pm}(\Sigma) \to 1 \tag{3}$$

where K_1 is the subgroup comprising isotopy classes of homeomorphisms preserving every fiber of $p^0 : M_1(\Sigma) \to \Sigma$.

Let us fix a principal $\mathbb{S}^1$-bundle structure on $p^0 : M_1(\Sigma) \to \Sigma$. For every f in K_1, and every x in Σ, the restriction of f to the fiber over x is a homeomorphism f_x, and the principal $\mathbb{S}^1$-bundle structure provides an identification of f_x as an element of $\mathrm{Homeo}(\mathbb{S}^1)$, well-defined up to conjugation by rotation.

Observe that every f_x preserves the orientation of the fiber. Indeed, if it was not the case, all f_x would be orientation reversing, and would admit 2 fixed points in every fiber. It would provide a section (maybe 2-multivalued) of p^0, that is, a contradiction.

Therefore, every f_x lies in $\mathrm{Homeo}^+(\mathbb{S}^1)$. In [Gh3, Proposition 4.2] E. Ghys defined a continuous retraction $\mathcal{R}$ from $\mathrm{Homeo}^+(\mathbb{S}^1)$ into $\mathbb{S}^1$, constant along classes of conjugation by rotation as follows: any element f of $\mathrm{Homeo}^+(\mathbb{S}^1)$ lifts as a homeomorphism $\tilde f$ of the real line $\mathbb{R}$ commuting with the translation $y \mapsto y+1$. Then $\mathcal{R}(f)$ is simply the integral $\int_0^1 (\tilde f(y) - y)dx$ mod 1. This map is a homotopy equivalence. Observe that $\mathcal{R}(f)$ commutes with the composition by rotations, hence invariant by conjugation by a rotation. It therefore provides a well-defined map $x \mapsto \mathcal{R}(f_x)$ that is continuous in x. The map f is homotopically equivalent to the homeomorphism of M that acts by rotation of angle $\mathcal{R}(f_x)$ in each fiber $[p^0]^{-1}(x)$, and since in closed Haken 3-manifolds homotopically equivalent homeomorphisms are isotopic ([Wald]), this homotopic equivalence is homotopic to an isotopy.

It follows that K_1 is the group of homotopy classes of maps from Σ into $\mathbb{S}^1$, which is notoriously isomorphic to $\mathrm{H}^1(\Sigma, \mathbb{Z})$. The exact sequence (3) becomes:

$$1 \to \mathrm{H}^1(\Sigma, \mathbb{Z}) \to \mathrm{Mod}(M_1(\Sigma)) \to \mathrm{Mod}^{\pm}(\Sigma) \to 1 \tag{4}$$

Remark 3.7. The way $\mathrm{H}^1(\Sigma, \mathbb{Z})$ acts on $M_1(\Sigma)$ can described as follows. For every α in $\mathrm{H}^1(\Sigma, \mathbb{Z})$, let ω be a closed 1-form representing α. Since α lies in $\mathrm{H}^1(\Sigma, \mathbb{Z})$, the periods $\int_c \omega$ for closed loops c are all integers. Select a point base x_0 in Σ, and for every x in Σ, let $R(x)$ be the element $\int_c \omega$ modulo $\mathbb{Z}$ of $\mathbb{R}/\mathbb{Z} \approx \mathbb{S}^1$, where c is any path from x_0 to x. Then let $f_\omega : M_1(\Sigma) \to M_1(\Sigma)$ be the map rotating every fiber $p^{-1}(x)$ by $R(x)$. The isotopy class of f_ω only depends on the cohomology class of ω, and represents the element $\alpha \in \mathrm{H}^1(\Sigma, \mathbb{Z}) \subset \mathrm{Mod}(M_1(\Sigma))$.

The induced action of $\mathrm{H}^1(\Sigma, \mathbb{Z})$ on $\widetilde\Gamma$ is described as follows: the action of $\alpha \in \mathrm{H}^1(\Sigma, \mathbb{Z})$ on $\widetilde\Gamma$ is simply the map sending every γ of $\widetilde\Gamma$ to $\gamma h^{\alpha(\bar\gamma)}$, where $\bar\gamma = p^0_*(\gamma)$. Here we use the canonical identification $\mathrm{H}^1(\Sigma, \mathbb{Z}) \approx \mathrm{H}^1(\bar\Gamma, \mathbb{Z})$.

A particular interesting case is the case of *vertical Dehn twists over a simple closed curve*: Let c be a simple closed oriented curve in Σ, and let U be a small collar neighborhood of c. The open domain $p^{-1}(U)$ is a tubular neighborhood of the torus $T := p^{-1}(c)$. Let ω_c be a closed 1-form in Σ, with support in U, and representing the cohomology class dual to the homology class of c: for every loop c', the integral $\int_{c'} \omega_c$ is the algebraic intersection number between c and c'. Then, the map f_{ω_c} is a vertical Dehn twist: it is the identity map outside $p^{-1}(U)$, and inside U, it rotates the fibers, more and more when we go from the left to the right of c, so that it adds to every homotopy class of curves crossing (positively) c a h-component.

Since cohomology classes dual to simple closed curves generate $\mathrm{H}^1(\Sigma,\mathbb{Z})$, vertical Dehn twists generate the kernel of the projection of $\mathrm{Mod}(M_1(\Sigma))$ onto $\mathrm{Mod}^{\pm}(\Sigma)$. Actually, vertical Dehn twists over the generators $\bar{a}_i$ and $\bar{b}_i$ of $\bar{\Gamma}$ are enough to generate this kernel.

Remark 3.8. The sequence (4) is split. Indeed, consider an element of $\mathrm{Mod}^{\pm}(\Sigma)$, represented by a homeomorphism f. Let $\tilde{f}$ be a lift of f in $\mathbb{H}^2 \approx \widetilde{\Sigma}$. This lift is well-defined up to composition by an element of $\bar{\Gamma}$. It extends to the conformal boundary $\partial\mathbb{H}^2 \approx \mathbb{R}P^1$ as a homeomorphism ∂f.

Let us first consider the case where f preserves the orientation, then its action on $\mathbb{R}P^1$ preserves the cyclic order, and there induces an action on the space X of triples $x < y < z$ introduced in Section 3.1.2. The induced action on the quotient of X by $\bar{\Gamma}$ does not depend on the choice of the lift $\tilde{f}$. Since this quotient is homeomorphic to $M_1(\Sigma)$, this process defines a morphism from $\mathrm{Mod}(\Sigma)$ into $\mathrm{Mod}(M_1(\Sigma))$. We denote by $\mathrm{Mod}(\Sigma)^o$ the image of this morphism. Observe that as it is defined, $\mathrm{Mod}(\Sigma)^o$ is a set of isotopy classes of homeomorphisms of $M_1(\Sigma)$, but actually we have realized it as a subgroup of genuine homeomorphisms of $M_1(\Sigma)$, in other words, we have selected an element in each isotopy class in a coherent way with respect to the group structure.

This construction does not apply when f is orientation reversing, since then for every element (x,y,z) of X the triple $(\partial f(x), \partial f(y), \partial f(z))$ does not belong to X since $\partial f(x) > \partial f(y) > \partial f(z)$. But it suffices to define the action of ∂f on X as given by:

$$\partial f.(x,y,z) = (\partial f(y), \partial f(x), \partial f(z))$$

Once again, the induced action on $\bar{\Gamma}\backslash X$ does not depend on the choice of the lift $\tilde{f}$, and this process defines a group homomorphism from $\mathrm{Mod}^{\pm}(\Sigma)$ into $\mathrm{Mod}(M_1(\Sigma))$, which is a section of the projection $\mathrm{Mod}(M_1(\Sigma)) \to \mathrm{Mod}^{\pm}(\Sigma)$. We denote by $\mathrm{Mod}^{\pm}(\Sigma)^o$ the image of this section.

Again observe that as it is defined, $\mathrm{Mod}^{\pm}(\Sigma)^o$ is a set of isotopy classes of homeomorphisms of $M_1(\Sigma)$, but actually we have realized it as a subgroup of genuine homeomorphisms of $M_1(\Sigma)$, in other words, we have selected an element in each isotopy class in a coherent way with respect to the group structure.

Remark 3.9. Observe that the homeomorphisms defined in Remark 3.8 preserve the orientation of X, in the case where f is orientation-preserving, but also in the case where it is orientation reversing. Indeed, in the last case, the homeomorphism is the restriction to X of the composition of the orientation reversing map $(x,y,z) \mapsto (\partial f(x), \partial f(y), \partial f(z))$ and the orientation reversing map $(x,y,z) \mapsto (y,x,z)$. On the other hand, the description in Remark 3.7 makes it clear that elements of K_1 are also orientation-preserving.

In view of the sequence (4), it follows that every element of $\mathrm{Mod}(M_1(\Sigma))$ is orientation-preserving.

Remark 3.10. This construction is closely related to the Baer-Dehn-Nielsen Theorem mentioned above. Actually, the circle $\mathbb{S}^1$ is naturally identified with the Gromov boundary $\partial\bar{\Gamma}$; hence the group of automorphisms $\mathrm{Aut}(\bar{\Gamma})$ acts on $\mathbb{S}^1$, and therefore on the space X of distinct triple of points in $\mathbb{S}^1$ satisfying $x < y < z$, if one performs as in Remark 3.8 in the orientation reversing case.

This action induces an action of $\mathrm{Out}(\bar{\Gamma})$ on the quotient of X by $\bar{\Gamma}$, i.e. a morphism from $\mathrm{Out}(\bar{\Gamma})$ into $\mathrm{Mod}(M_1(\Sigma))$. Composing this morphism with the projection induces a morphism $\mathrm{Out}(\bar{\Gamma}) \to \mathrm{Mod}^{\pm}(\Sigma)$ which is precisely the Baer-Dehn-Nielsen isomorphism. Observe that there is an obvious inverse of this map: the one associating to an element of $\mathrm{Mod}^{\pm}(\Sigma)$ its induced action on the fundamental group $\bar{\Gamma}$.

Recall that $\mathfrak{I}_1$ is the set of isotopy classes of Anosov flows on $M_1(\Sigma)$. The modular group $\mathrm{Mod}(M_1(\Sigma))$ acts clearly on $\mathfrak{I}_1$ since every element of $\mathrm{Mod}(M_1(\Sigma))$ admits a smooth representative (but there is no way to choose simultaneously such representatives for every element of $\mathrm{Mod}(M_1(\Sigma))$, see [Sou]).

Proposition 3.11. *The action of* $\mathrm{Mod}(M_1(\Sigma))$ *on* $\mathfrak{I}_1$ *is transitive, and the stabilizer of the isotopy class of the geodesic flow* Φ_0 *is the subgroup* $\mathrm{Mod}^{\pm}(\Sigma)^o$ *defined in Remark 3.8.*

Proof. The transitivity property of the action comes from Theorem 3.3: this theorem gives us a finite cover $p : M_1(\Sigma) \mapsto M_1(\Sigma)$ satisfying the conclusion of Theorem 3.3. Using the Euler class of the bundle $M_1(\Sigma)$ it follows that the cover is of degree one, hence a homeomorphism, so p is in $\mathrm{Mod}(M_1(\Sigma))$. Transitivity follows.

We now analyze the stabilizer. Denote by $[\Phi_0]$ the isotopy class of the geodesic flow on $M_1(\Sigma)$, and by Φ_0 the representative of $[\Phi_0]$ induced by an arbitrary but fixed hyperbolic metric g_0 on Σ. We fix Φ_0 throughout the proof.

Let now η be an element of $\mathrm{Mod}^{\pm}(\Sigma)$. Consider as in Remark 3.8 the action ∂f on the circle $\partial\mathbb{H}^2 \approx \mathbb{R}P^1$ of some lift $\tilde{f}$ of a representative f of η (hence $\eta = [f]$). Remark 3.8 is based on the observation that the transformation

$$(x, y, z) \ \mapsto \ (\partial f(x), \partial f(y), \partial f(z))$$

when f is orientation-preserving, and

$$(x, y, z) \ \mapsto \ (\partial f(y), \partial f(x), \partial f(z))$$

when f is orientation reversing, is $\bar{\Gamma}$-equivariant on X. Therefore this transformation induces a homeomorphism F on $\bar{\Gamma}\backslash X \approx M_1(\Sigma)$. This homeomorphism F preserves the *non-oriented* foliation $x = Cte$, $y = Cte$. As explained at the end of Section 3.1, this foliation is a representative of the *non-oriented* foliation induced by the geodesic flow. Therefore, $F_*\Phi_0$ is isotopic to the geodesic flow or to its inverse. But the geodesic flow is isotopic to its own inverse, see the observation just after Theorem 2.9. Hence, in all cases, $F_*\Phi_0$ is isotopic to the geodesic flow. We conclude that $\mathrm{Mod}^{\pm}(\Sigma)^o$ is contained in the stabilizer of $[\Phi_0]$.

Since the sequence (4) is split it follows that the group $\mathrm{H}^1(\Sigma, \mathbb{Z}) \approx \mathrm{Mod}(M_1(\Sigma))/\mathrm{Mod}^{\pm}(\Sigma)^o$ acts transitively on $\mathfrak{I}_1$. In order to conclude, we just have to show that non-trivial elements of $\mathrm{H}^1(\Sigma, \mathbb{Z})$ do not belong to the stabilizer of $[\Phi_0]$.

Let α be an element of $\mathrm{H}^1(\Sigma, \mathbb{Z})$, and let $\bar{\gamma}$ be a non-trivial element of $\bar{\Gamma}$. The action of $\bar{\Gamma}$ on $\mathbb{H}^2 \approx \widetilde{\Sigma}$ preserves a unique geodesic (xy) in $\mathbb{H}^2$, with x, $y \in \partial\mathbb{H}^2$, so that $\bar{\gamma}$ acts on this geodesic as a translation from y towards x. The vectors in $T^1\mathbb{H}^2$ tangent to this geodesic and pointing in the direction of x form an orbit of the geodesic flow of $\mathbb{H}^2$.

This geodesic lifts in the universal covering of $M_1(\Sigma)$ to infinitely many orbits of $\widetilde{\Phi}_0^t$ that are permuted under the action of the fiber h, and there is one and only one element $\gamma = \sigma_0(\bar{\gamma})$ of $(p_*^0)^{-1}(\bar{\gamma})$ preserving each of these lifted geodesics (see Remark 3.1): all other elements of $(p_*^0)^{-1}(\bar{\Gamma})$ have the form γh^k for some integer k, and when $k \neq 0$, γh^k preserves no orbit of $\widetilde{\Phi}_0^t$.

Now apply the element $\alpha \in \mathrm{H}^1(\Sigma, \mathbb{Z}) \subset \mathrm{Mod}(M_1(\Sigma))$ to the geodesic flow Φ_0^t, and assume that $\alpha_*\Phi_0^t$ is isotopic to Φ_0^t. The homotopy class γ is mapped under α to the homotopy class $\gamma h^{\alpha(\bar{\gamma})}$. Since $\alpha_*\Phi_0^t$ is isotopic to Φ_0^t, this homotopy class must preserve an orbit of $\widetilde{\Phi}_0^t$, hence we must have $\alpha(\bar{\gamma}) = 0$. Therefore, since $\bar{\gamma}$ is arbitrary, α is trivial. The proposition follows. □

In summary, there are infinitely many isotopy classes of Anosov flows on $M_1(\Sigma)$ parametrized by $\mathrm{Mod}^{\pm}(M_1(\Sigma))/\mathrm{Mod}^{\pm}(\Sigma)^0 \approx \mathrm{H}^1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^{2g}$. It follows that isotopy classes can be distinguished according to periodic orbits over the generators $\bar{a}_i$ and $\bar{b}_i$ of $\bar{\Gamma}$: if two Anosov flows on $M_1(\Sigma)$ have the property that the periodic orbits "above the generators $\bar{a}_i$ and $\bar{b}_i$", for all i, are freely homotopic for the two flows, then the two flows are isotopic.

3.3. **Isotopy classes of Anosov flows on circle bundles over closed surfaces.** In all this subsection, M is an oriented 3-manifold, admitting a fibration by circles $p : M \to \Sigma$ over a closed, connected, oriented hyperbolic surface Σ. We will denote by Θ_0 the foliation whose leaves are the fibers of p. Observe that this foliation is oriented, is a foliation by circles, and is unique up to isotopy.

This section is devoted to the study of isotopy classes of Anosov flows on M. In the next subsection we will study orbit equivalence classes of Anosov flows. In the next section we prove the main theorem of this paper (Theorem 4.5).

According to Theorem 3.3, up to reparametrization, every Anosov flow on M is the pull-back of the geodesic flow Φ_0^t by a finite covering $q : M \to M_1(\Sigma)$. Moreover, $M_1(\Sigma)$ is precisely the unit tangent bundle over the surface Σ over which M is assumed to be fibering. It implies that the finite covering $q : M \to M_1(\Sigma)$ is *a covering along the fibers*, meaning that p is isotopic to the composition $p^0 \circ q$. In other words, the pull-back by q of the fibration p^0 is isotopic to the fibration induced by p. This again is because the Seifert fibration in M is unique up to isotopy. Modifying q by this isotopy (notice that this does not modify the isotopy class of the Anosov foliation in M), one can always assume that the equality $p = p^0 \circ q$ holds.

In other words, a covering $N \to M_1(\Sigma)$ is not along the fibers if the base surface of the Seifert fibering of N is not homeomorphic to Σ, or, alternatively, if the preimages of fibers of $M_1(\Sigma)$ by the covering map are not connected.

All these finite coverings $q : M \to M_1(\Sigma)$ have the same degree n: it is the quotient $(2g-2)/|d|$, where $|d|$ is the absolute value of the Euler class d of $p : M \to \Sigma$.

This remark makes also clear that all the finite covers $M_1(\Sigma)$ of degree n along fibers are all homeomorphic one to the others: there are all circle bundles over Σ of Euler class $(2g-2)/n$.

We propose now a useful alternative definition of finite coverings along the fibers. Every leaf of Θ_0, i.e. every fiber of p, is an oriented circle. One can therefore define the homeomorphism g_0 mapping every x in M to the next element of the leaf of Θ_0 that maps to the same point as x by p, according to the orientation of this fiber. This homeomorphism has order n and induces a free action of $\mathbb{Z}/n\mathbb{Z}$ on M.

More generally, let Θ be an oriented foliation of M by circles, and let g be a homeomorphism of M of order n, preserving every leaf of Θ and such that the action of $\mathbb{Z}/n\mathbb{Z}$ on M induced by g is free. Then, we say that g is Θ-*increasing* if:

– either $n = 1$ or 2,

– or $n \geq 2$ and for every x in M the triple $(x, g(x), g^2(x))$ is positive for the cyclic order of the (oriented) leaf of Θ containing x.

Lemma 3.12. *There is an one-to-one correspondence between:*

- *Finite coverings $q : M \to M_1(\Sigma)$ of degree n along the fibers,*
- *The data of an oriented foliation Θ of M by circles, a Θ-increasing homeomorphism g of M of order n without fixed points preserving every leaf of Θ, and an homeomorphism between $M_1(\Sigma)$ and the orbit space of the $\mathbb{Z}/n\mathbb{Z}$-action in M.*

The proof of this Lemma is quite straightforward, once observed that the only foliations of M by circles are the ones induced by circle fibrations of M over Σ. This is because any foliation by circles in M is a Seifert fibration [Ep], then Seifert fibrations in M are unique up to isotopy, and finally there is a circle fibration of M over Σ. We just point out that given the free $\mathbb{Z}/n\mathbb{Z}$-action and the homeomorphism Ψ between the orbit space N of the $\mathbb{Z}/n\mathbb{Z}$-action and $M_1(\Sigma)$, the finite covering is simply $\Psi \circ \varrho$, where $\varrho : M \to N$ is the projection to the quotient space. The other direction is straightforward.

After all these preliminaries, we go back to the problem of the study of isotopy classes of Anosov flows on M. Once again, according to Theorem 3.3, it leads to the study of the set $\mathfrak{Cov}_n^{\pm}$ of finite coverings of degree n along the fibers up to the following equivalence relation: two such covering maps $q_1 : M \to M_1(\Sigma)$ and $q_2 : M \to M_1(\Sigma)$ are *isotopic* if there exists a homotopically trivial homeomorphism $F : M \to M$ such that $q_2 = q_1 \circ F$. Indeed, if q_1 and q_2 are isotopic in this sense, then the Anosov flows $q_1^*\Phi_0$ and $q_2^*\Phi_0$ are isotopic in M.

Any homotopically trivial homeomorphism of $M_1(\Sigma)$ lifts to a homotopically trivial homeomorphism of M, therefore the mapping class group $\mathrm{Mod}(M_1(\Sigma))$ acts on $\mathfrak{Cov}_n^{\pm}$ by composition on the left. The $\pm$ refers to covers preserving or reversing orientation.

Proposition 3.13. *The action of* $\mathrm{Mod}(M_1(\Sigma))$ *on* $\mathfrak{Cov}_n^{\pm}$ *is simply transitive.*

Proof. First observe that this action is free. Indeed: let f be a homeomorphism of $M_1(\Sigma)$ and $q_1 : M \to M_1(\Sigma)$ be a finite covering such that there is a homotopically trivial homeomorphism F of M such that $f \circ q_1 = q_1 \circ F$. Then, f must be homotopically trivial too, and therefore trivial in $\mathrm{Mod}(M_1(\Sigma))$.

Let now $q_0 : M \to M_1(\Sigma)$ and $q : M \to M_1(\Sigma)$ be two covering maps of degree n along the fibers. We want to show that q is isotopic to a finite covering of the form $f \circ q_0$ for some homeomorphism f of $M_1(\Sigma)$.

According to Lemma 3.12, one can interpret q_0 and q as the data of two foliations by circles Θ_0 and Θ on M, two free actions of $\mathbb{Z}/n\mathbb{Z}$ induced by increasing homeomorphisms g_0 and g on M, the first preserving Θ_0 leafwise and the second preserving Θ leafwise, and identifications $f_0 : N_0 \to M_1(\Sigma)$ and $f : N \to M_1(\Sigma)$, where N_0 is the orbit space of the first $\mathbb{Z}/n\mathbb{Z}$-action, and N the orbit space of the second $\mathbb{Z}/n\mathbb{Z}$-action.

Up to isotopy, one can assume that the circle foliations Θ_0 and Θ coincide. Then, in every leaf of $\Theta = \Theta_0$, the two $\mathbb{Z}/n\mathbb{Z}$-actions are both conjugated (through a homeomorphism isotopic to the identity)

to the same action by rotations, meaning that they are conjugated one to the other. These conjugacies in all the leaves of Θ can be combined to some continuous conjugacy F between the two $\mathbb{Z}/n\mathbb{Z}$-actions preserving every leaf of Θ_0. This conjugacy is not necessarily isotopic to the identity.

Let us fix a principal $\mathbb{R}/\mathbb{Z}$-structure on $p^0 : M_1(\Sigma) \to \Sigma$. The pull-back of principal $\mathbb{R}/\mathbb{Z}$-structure by the finite covering q_0 is a principal $\mathbb{R}/n\mathbb{Z}$-structure on $p : M \to \Sigma$. Then, for every x in M and every $a \in \mathbb{R}$ the image by q_0 of $a.x$ is equal to $a.q_0(x)$. There is a map $f : \Sigma \to \mathrm{Homeo}^+(\mathbb{R}/n\mathbb{Z})$ such that, for every x in M, the image $F(x)$ is the image of x under $f(p(x)) \in \mathrm{Homeo}^+(\mathbb{R}/n\mathbb{Z})$.

We apply Ghys' argument as we did just after Corollary 3.6: up to isotopy on F, one can assume that every $f(x)$ for x in Σ is a rotation. In other words, for every x in M we have $F(x) = f(p(x)).x$, where $f(p(x))$ in an element of $\mathbb{R}/n\mathbb{Z}$ – here we are now identifying the rotation number with the rotation homeomorphism associated with that rotation number. Let $[f(p(x)]$ be the projection of $f(p(x)) \in \mathbb{R}/n\mathbb{Z}$ in $\mathbb{R}/\mathbb{Z}$. Then, the map $x \mapsto [f(p(x)]$ defines a vertical Dehn twist f_ω as described in Remark 3.7.

Therefore, up to replacing q_0 by $f_\omega \circ q_0$ where $f_\omega \in K_1 \subset \mathrm{Mod}(M_1(\Sigma))$, one can assume that both $\mathbb{Z}/n\mathbb{Z}$-actions coincide.

Therefore, the orbit spaces N and N_0 do coincide, and the projection maps $\varrho_0 : M \to N_0$ and $\varrho : M \to N$ as well.

The last step is to observe that the homeomorphisms $f_0 : N_0 \to M_1(\Sigma)$ and $f : N_0 \to M_1(\Sigma)$ do not necessarily coincide, even if $N = N_0$. However, it means that q coincides with $g \circ q_0$, where $g = f \circ f_0^{-1}$, meaning as required that the isotopy class of q is the image of the isotopy class of q_0 under $[g]$. The Proposition is proved. □

Remark 3.14. Since every element of $\mathrm{Mod}^{\pm}(M_1(\Sigma)) = \mathrm{Mod}(M_1(\Sigma))$ is orientation-preserving (Remark 3.9) it follows from Proposition 3.13 that every finite covering over $M_1(\Sigma)$ is orientation-preserving, i.e. that $\mathfrak{Cov}_n^{\pm} = \mathfrak{Cov}_n$.

Let $\mathfrak{I}_n$ be the space of isotopy classes of Anosov flows in M. According to Theorem 3.3, every Anosov foliation on M is the pull-back $q^*\Phi_0$ of the geodesic foliation Φ_0 by a finite covering $q : M \to M_1(\Sigma)$. It means that the map $\# : \mathfrak{Cov}_n \to \mathfrak{I}_n$ associating to (the isotopy class of) a finite covering $q : M \to M_1(\Sigma)$ the (the isotopy class of the) pull-back $q^*\Phi_0$ of the geodesic flow is surjective.

Corollary 3.15. *The fibers of the map $\# : \mathfrak{Cov}_n \to \mathfrak{I}_n$ are precisely the $\mathrm{Mod}^{\pm}(\Sigma)^0$-orbits. In particular, there is a one-to-one correspondence between $\mathfrak{I}_n$ and $K_1 \approx \mathrm{H}^1(\bar{\Gamma}, \mathbb{Z})$.*

Proof. We use Proposition 3.13. Since $\mathrm{Mod}^{\pm}(\Sigma)^0$ is the stabilizer of $[\Phi_0]$, the map $\# : \mathfrak{Cov}_n \to \mathfrak{I}_n$ is constant along $\mathrm{Mod}^{\pm}(\Sigma)^0$-orbits. Since the exact sequence (4) is split, it follows that the restriction of $\#$ to any K_1-orbit is still surjective. The Corollary will be proved if we show that this restriction to some K_1-orbit is injective. We will show it by using the same argument as in the end of the proof of Proposition 3.11.

More precisely: let us consider the K_1-orbit of the preferred finite covering $q_0 : M \to M_1(\Sigma)$ we have selected. Let α be a non trivial element of $K_1 \approx \mathrm{H}^1(\bar{\Gamma}, \mathbb{Z})$. Let q be the finite covering obtained by composing the vertical twist associated to α and q_0. Assume that $q_0^*\Phi_0$ and $q^*\Phi_0$ are isotopic in M. But since $q = \alpha \circ q_0$ then $q^* = q_0^* \circ \alpha^*$.

Now consider Φ_0 and $\Phi_1 = \alpha^*(\Phi_0)$. Since α is in K_1 it follows that Φ_0 and Φ_1 are not isotopic in $M_1(\Sigma)$. In fact, according to the discussion at the end of Section 3.2, there is some simple closed geodesic θ in Σ representing some non-trivial element $\bar{\gamma}$ of $\bar{\Gamma}$ such that the periodic orbits θ_0, θ_1 of Φ_0, Φ_1 respectively above θ are not isotopic in $M_1(\Sigma)$. It follows that the periodic orbits of $q_0^*\Phi_0$ and $q_0^*\Phi_1$ above θ are not isotopic. But

$$q_0^*\Phi_1 = (\alpha \circ q_0)^*\Phi_0 = q^*\Phi_0.$$

By hypothesis $q_0^*\Phi_0$ and $q^*\Phi_0$ are isotopic flows. Hence these periodic orbits should all be isotopic to each other, since the periodic orbits in the the torus above θ are all isotopic to each other. This is a contradiction.

The corollary is proved. □

3.4. **Anosov flows on circle bundles over closed surfaces up to orbital equivalence.** In this subsection, we still denote by M a fixed closed orientable circle bundle, which admits a finite covering of

degree n along the fibers over $M_1(\Sigma)$. We will study orbital equivalences of Anosovflows on M, i.e. finite coverings of geodesic flows. For this purpose, we change our point of view and consider the geodesic flow as an algebraic flow (cf. Section 3.1.1). Hence finite coverings can be described as quotients of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ by a subgroup Γ of index n in the lattice $\widetilde{\Gamma}$.

Definition 3.16. *Let $\mathfrak{G}_n$ be the set of subgroups Γ of $\widetilde{\Gamma}$ such that:*
– *the restriction of $p_*^0 : \widetilde{\Gamma} \to \bar{\Gamma}$ to Γ is surjective,*
– *Γ has finite index n in $\widetilde{\Gamma}$*

In other words, $\mathfrak{G}_n$ is the set of subgroups of $\widetilde{\Gamma}$ corresponding to finite coverings of degree n along the fibers. Observe that for any element Γ of $\mathfrak{G}_n$, the center $Z(\Gamma)$ is the intersection between Γ and the center of $\widetilde{\Gamma}$. More precisely, if h denotes a generator of the center of $\widetilde{\Gamma}$ (corresponding to fibers of $M_1(\Sigma)$), the center is generated by h^n. Therefore, the first condition in Definition 3.16 means that the quotient group $\Gamma/Z(\Gamma)$ is isomorphic to $\bar{\Gamma}$.

According to Ghys's Theorem, any Anosov flow on M is orbitally equivalent to the flow on $\Gamma\backslash\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ induced by the right action of the diagonal group D for some element Γ of $\mathfrak{G}_n$ (see subsection 3.1.1). We will denote by Φ_Γ this flow.

The key observation we want to start with is that $\mathrm{Mod}(M_1(\Sigma))$ acts naturally on $\mathfrak{G}_n$. Indeed: let $[F]$ be a element of $\mathrm{Mod}(M_1(\Sigma))$, represented by some homeomorphism F on $M_1(\Sigma)$. Let F_* be the action on $\pi_1(M_1(\Sigma)) \approx \widetilde{\Gamma}$. This automorphism of $\widetilde{\Gamma}$ is defined up to inner automorphism, but we claim that for any element Γ of $\mathfrak{G}_n$, the subgroup $F_*(\Gamma)$ does not depend on this choice.

Indeed: let $\tilde{\gamma}$ be any element of $\widetilde{\Gamma}$. Then, according to the first property in Definition 3.16, there is an element γ of Γ and an integer k such that $\tilde{\gamma} = \gamma h^k$. Then, since h lies in the center of $\widetilde{\Gamma}$, the conjugate of Γ under $\tilde{\gamma}$ is equal to its conjugate by γ, and therefore itself since γ lies in Γ. The claim is proved.

Moreover, since F_* must preserves the kernel of p_*^0 (because it is the center of $\widetilde{\Gamma}$), the image $F_*(\Gamma)$ is still an element of $\mathfrak{G}_n$. We have proved our first key observation.

Therefore, the subgroup $\mathrm{Mod}^{\pm}(\Sigma)^0$ acts naturally on $\mathfrak{G}_n$.

More precisely, let φ be an automorphism of $\bar{\Gamma}$, representing an element $[\varphi]$ of $\mathrm{Out}(\bar{\Gamma}) \approx \mathrm{Mod}^{\pm}(\Sigma)$, therefore an element of $\mathrm{Mod}^{\pm}(\Sigma)^0$. As explained in Remarks 3.8, 3.9, φ induces an homeomorphism F in $M_1(\Sigma)$, which is an self orbital equivalence of the geodesic flow. Therefore, φ maps every element Γ of $\mathfrak{G}_n$ to the subgroup $F_*(\Gamma)$. Now, for every element γ of Γ let $f_*(\gamma)$ be the image under p_*^0 of $F_*(\gamma)$. If γ' is another element of Γ admitting the same projection in $\bar{\Gamma}$ than γ, there is some integer k such that $\gamma' = h^k\gamma$. Then:

$$F_*(\gamma') = F_*(\gamma)F_*(h)^k = F_*(\gamma)h^{\pm k}.$$

Therefore, $f_*(\gamma') = f_*(\gamma)$, and actually since the projection by p_*^0 restricted to Γ and $F_*(\Gamma)$ as well is surjective, f_* defines an automorphism of $\bar{\Gamma}$ into itself, well-defined up to inner automorphims. Actually, it follows from our construction that this automorphism, up to some inner automorphism, coincides with the initial automorphism φ.

We can now state the main result of this subsection:

Proposition 3.17. *Let Γ, Γ' be two elements of $\mathfrak{G}_n$. Then, Φ_Γ and $\Phi_{\Gamma'}$ are orbitally equivalent if and only if Γ and Γ' lie in the same orbit under the action of $\mathrm{Mod}^{\pm}(\Sigma)^0$ on $\mathfrak{G}_n$.*

Proof. Since the homeomorphisms of $M_1(\Sigma)$ send a geodesic flow foliation to a geodesic flow foliation, it is quite clear that if Γ and Γ' lie on the same $\mathrm{Mod}^{\pm}(\Sigma)^0$-orbit, the associated Anosov flows are orbitally equivalent - there is a homeomorphism between $\Gamma\backslash\widetilde{M}$ and $\Gamma'\backslash\widetilde{M}$ mapping the first finite covering of the geodesic flow to a flow isotopic to the second finite covering of the geodesic flow.

Assume now that the Anosov flows associated to Γ and Γ' are orbitally equivalent. It means that there is an orbital equivalence between $\widetilde{\Phi}$ (which is the lift to the universal cover of the geodesic flow on the surface) and itself. This induces a homeomorphism $F : \mathcal{O} \to \mathcal{O}$ on the orbit space.

Furthermore, this orbital equivalence preserves the stable and unstable foliations, and therefore induces homeomorphisms F^s and F^u of the leaf spaces Q^s and Q^u, respectively.

The orbital equivalence induces a morphism $F_* : \Gamma \to \Gamma'$, such that, for every element γ of Γ:

$$F \circ \gamma \quad = \quad F_*(\gamma) \circ F \tag{5}$$

Here we are abusing notation and also denoting by F a lift of the orbital equivalence in question. Similarly:

$$\begin{aligned} F^s \circ \gamma &= F_*(\gamma) \circ F^s \\ F^u \circ \gamma &= F_*(\gamma) \circ F^u \end{aligned}$$

According to the discussion just before the statement of Proposition 3.17, if we replace Γ' by some element of its $\mathrm{Mod}^{\pm}(\Sigma)^0$-orbit, one can assume that F_* induces a trivial element of $\mathrm{Out}(\bar\Gamma)$. Therefore, there exists an element γ_0 of Γ and a map $\alpha : \Gamma \to \mathbb{Z}$ such that:

$$\forall \gamma \in \Gamma \;\; F_*(\gamma) = h^{\alpha(\gamma)} \gamma_0 \gamma \gamma_0^{-1}.$$

Replace F by $F \circ \gamma_0^{-1}$, which is still an orbital equivalence between $\widetilde\Phi$ and itself. Then:

$$\forall \gamma \in \Gamma \;\; F_*(\gamma) = h^{\alpha(\gamma)} \gamma.$$

It follows that $\alpha : \Gamma \to \mathbb{Z}$ is a group homomorphism.

Recall (Section 3.1.3) that in the case of the geodesic flow there is an identification between $\mathcal{O}$ and the region Ω_0 of $\widetilde{\mathbb{R}P^1} \times \widetilde{\mathbb{R}P^1}$ between the graphs of the identity map and the graph of δ, where δ generates the center of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. Then, there is some homeomorphism $\bar F : \widetilde{\mathbb{R}P^1} \to \widetilde{\mathbb{R}P^1}$ (corresponding to F^s or F^u when Q^s or Q^u is identified with $\widetilde{\mathbb{R}P^1}$) so that F corresponds to the map $(x,y) \mapsto (\bar F(x), \bar F(y))$. Here we are using the identification of Q^u with Q^s via the map β^{-1}, see Remark 2.5. Since this diagonal map must preserve Ω_0, it follows that for every x in $\widetilde{\mathbb{R}P^1}$ we have $(\bar F(x), \bar F(\delta x)) = (\bar F(x), \delta \bar F(x))$, hence $\bar F \circ \delta = \delta \circ \bar F$.

It follows that $\bar F$ induces a map g on $\mathbb{R}P^1$. Equation (5) and the equation $F_*(\gamma) = h^{\alpha(\gamma)}\gamma$ above imply that the induced map g in $\mathbb{R}P^1$ commutes with the action of $\bar\Gamma$ on $\mathbb{R}P^1$. Since fixed points of elements of $\bar\Gamma$ in $\mathbb{R}P^1$ are dense, such a map g is necessarily the identity map of $\mathbb{R}P^1$. It follows that $\bar F$ is equal to some power δ^k. In other words, the map F coincides with the action on $\mathcal{O}$ of some power h^k. Equation (5) implies that F_* is trivial, that is F_* is the identity map. Hence the morphism α is trivial, and Γ' coincides with Γ.

The Proposition is proved. □

4. Counting the number of orbits of the action of the modular group on the set of finite index subgroups

It follows from Proposition 3.17 that the description of orbital equivalence classes of Anosov foliations in M reduces to the description of the action of $\mathrm{Mod}^{\pm}(\Sigma)^0 \approx \mathrm{Mod}^{\pm}(\Sigma)$ on $\mathfrak{G}_n$. For that purpose, in this section, we provide a convenient parametrization of $\mathfrak{G}_n$ by $\mathrm{H}^1(\bar\Gamma, \mathbb{Z}/n\mathbb{Z})$, so that the action of the modular group appears as preserving the natural affine structure of $\mathrm{H}^1(\bar\Gamma, \mathbb{Z}/n\mathbb{Z})$.

Let Γ, Γ' be two elements of $\mathfrak{G}_n$. The restrictions of $p_*^0 : \widetilde\Gamma \to \bar\Gamma$ to Γ and Γ' are both surjective, with kernel the subgroup $n\mathbb{Z}$ generated by h^n of the center $\mathbb{Z}$ of $\widetilde\Gamma$. For every element γ of Γ, there is some integer $\alpha(\gamma)$ such that $h^{\alpha(\gamma)}\gamma$ lies in Γ'. More precisely, two elements of Γ' (or Γ) project on the same element of $\bar\Gamma$ if and only if their "difference" is an iterate of h^n.

Therefore, the integer $\alpha(\gamma)$ is unique modulo n, and only depends on the projection $p_*^0(\gamma)$. It defines an application

$$\bar\alpha : \bar\Gamma \to \mathbb{Z}/n\mathbb{Z}$$

and it is easy to check that this application is a morphism. This element of $\mathrm{H}^1(\bar\Gamma, \mathbb{Z}/n\mathbb{Z})$ can be seen as the difference between the two elements Γ and Γ'.

Now fix Γ and let Γ' vary in $\mathfrak{G}_n$. Each Γ' in $\mathfrak{G}_n$ has an associated morphism $\bar\alpha$, which for simplicity of notation we omit the dependence of $\bar\alpha$ on Γ'. We study the association $\Gamma' \to \bar\alpha$. For any Γ' in $\mathfrak{G}_n$, and for any $\gamma_1', \gamma_2' \in \Gamma'$ with $p_*^0(\gamma_1') = p_*^0(\gamma_2')$ then $\gamma_2' = h^{ni}\gamma_1'$ for some i integer, and for any i integer, such γ_2' is in Γ'. It follows that Γ' is determined by $\bar\alpha$, that is, the association Γ' to its $\bar\alpha$ function is injective. On the other hand given a morphism $\bar\alpha : \bar\Gamma \to \mathbb{Z}/n\mathbb{Z}$, one builds a subset Γ' of $\tilde\Gamma$ as follows: for each $\bar\gamma$ in $\bar\Gamma$ choose γ in Γ with $p_*^0(\gamma) = \bar\gamma$ and consider all $\gamma' \in \tilde\Gamma$ so that $\gamma' = h^{a+ni}\gamma$, where i is any integer and a

is a fixed integer so that $a = \bar{\alpha}(\bar{\gamma})\ mod\ n$. It is easy to see that Γ' is a subgroup of $\tilde{\Gamma}$ (using that $\bar{\alpha}$ is a morphism), that it has index n, and that $p_*^0(\Gamma') = \bar{\Gamma}$. It follows that the association Γ' to $\bar{\alpha}$ is surjective, so in fact this association is a bijection.

As a corollary, it is easy to see that $\mathfrak{G}_n$ is an affine space with underlying $\mathbb{Z}/n\mathbb{Z}$-module $\mathrm{H}^1(\bar{\Gamma}, \mathbb{Z}/n\mathbb{Z})$ (which is a genuine vector space when n is prime for example), and it follows from the discussion just before Proposition 3.17 that the action of $\mathrm{Mod}^{\pm}(\Sigma)^0$ on $\mathfrak{G}_n$ preserves this affine structure. It is an affine structure because there is no obvious origin − recall that we picked an element $\Gamma \in \mathfrak{G}_n$ to start with.

Remark 4.1. There is another way to express this identification between $\mathfrak{G}_n$ and $\mathrm{H}^1(\bar{\Gamma}, \mathbb{Z}/n\mathbb{Z})$. Let $\hat{\Gamma}_n$ be the quotient of $\widetilde{\Gamma}$ by the normal subgroup generated by h^n. There is an exact sequence:

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \hat{\Gamma}_n \to \bar{\Gamma} \to 1 \tag{6}$$

which now is split since n divides the Euler class. Then, the projection in $\hat{\Gamma}_n$ of an element Γ of $\mathfrak{G}_n$ is a subgroup whose projection to $\bar{\Gamma}$ is an isomorphism. Hence, it provides a splitting of (6), and this provides a one-to-one correspondance between $\mathfrak{G}_n$ and the set of splittings of (6). This last one is notoriously $\mathrm{H}^1(\bar{\Gamma}, \mathbb{Z}/n\mathbb{Z})$.

We start with still another way to describe elements of $\mathfrak{G}_n$, but this time well-suited for describing the action of $\mathrm{Mod}^{\pm}(\Sigma)$ on $\mathfrak{G}_n$.

Consider the section $\sigma_0 : \bar{\Gamma} \to \widetilde{\Gamma}$ defined in Remark 3.1. We recall that it is the map associating to every element of $\bar{\gamma}$ in $\bar{\Gamma}$ the element $\widetilde{\gamma}$ in $(p_*^0)^{-1}(\bar{\gamma})$ preserving an orbit of the lifted geodesic flow. For this, it is good to realize $\bar{\Gamma}$ as an uniform lattice in $\mathrm{PSL}(2, \mathbb{R})$, and $\widetilde{\Gamma}$ as a lattice in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, acting on $\widetilde{\mathbb{R}P^1}$. For every $\bar{\gamma}$ in $\bar{\Gamma}$, $\sigma_0(\bar{\gamma})$ is the unique element in $\widetilde{\Gamma}$ above $\bar{\gamma}$ admitting a fixed point in $\widetilde{\mathbb{R}P^1}$.

The section σ_0 is not a morphism, recall Remark 3.1 (the difference between h and δ is simply that the first generates the center of $\widetilde{\Gamma}$ whereas δ generates the center of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$; they coincide once $\widetilde{\Gamma}$ identified as a lattice in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$). Therefore

$$\sigma_0(\bar{\gamma}_1\bar{\gamma}_2) = h^{c(\bar{\gamma}_1, \bar{\gamma}_2)}\sigma_0(\bar{\gamma}_1)\sigma_0(\bar{\gamma}_2) \tag{7}$$

where c is the Euler cocycle.

For every $\bar{\gamma}$ in $\bar{\Gamma}$ the element $\sigma_0(\bar{\gamma})^{-1}$ preserves the same orbit than $\sigma_0(\bar{\gamma})$, hence:

$$\sigma_0(\bar{\gamma}^{-1}) = \sigma_0(\bar{\gamma})^{-1} \tag{8}$$

Moreover, σ_0 is equivariant with respect to inner automorphisms: for every γ_0 in $\widetilde{\Gamma}$ we have:

$$\sigma_0(\bar{\gamma}_0\bar{\gamma}\bar{\gamma}_0^{-1}) = \gamma_0\sigma_0(\bar{\gamma})\gamma_0^{-1} \tag{9}$$

where $\bar{\gamma}_0 = p_*^0(\gamma_0)$. Indeed, the conjugate $\gamma_0\sigma_0(\bar{\gamma})\gamma_0^{-1}$ preserves the image under γ_0 of the orbit preserved by $\sigma_0(\bar{\gamma})$.

Let us now see how to parametrize $\mathfrak{G}_n$. Let $\sigma : \bar{\Gamma} \to \hat{\Gamma}_n$ be a morphism representing an element of $\mathfrak{G}_n$ (cf. Remark 4.1). Then, there is a map $\nu : \bar{\Gamma} \to \mathbb{Z}/n\mathbb{Z}$ defined by:

$$\sigma(\bar{\gamma}) = \bar{\sigma}_0(\bar{\gamma})\bar{h}^{\nu(\bar{\gamma})}$$

where $\bar{\sigma}_0(\bar{\gamma})$ and $\bar{h}$ are the projections in $\hat{\Gamma}_n$ of $\sigma_0(\bar{\gamma})$ and h.

As was the case for the map $\bar{\alpha}$ it follows that the association $\Gamma' \in \mathfrak{G}_n$ to ν is injective. The map ν is more complicated than the map $\bar{\alpha}$ in the following ways:

First, the map ν is not a morphism, but must satisfy the following equation, for every pair $(\bar{\gamma}_1, \bar{\gamma}_2)$ of elements of $\bar{\Gamma}$:

$$\nu(\bar{\gamma}_1\bar{\gamma}_2) = \nu(\bar{\gamma}_1) + \nu(\bar{\gamma}_2) - c(\bar{\gamma}_1, \bar{\gamma}_2), \tag{10}$$

where the equation should be understood in $\mathbb{Z}/n\mathbb{Z}$. In other words, we are equating $c(\bar{\gamma}_1, \bar{\gamma}_2)$ which is an integer, with its projection in $\mathbb{Z}/n\mathbb{Z}$. The equation above means that the coboundary of the 1-cochain ν is the 2-cocycle c: the Euler class of the surface with coefficients in $\mathbb{Z}/n\mathbb{Z}$ is indeed trivial, i.e. a coboundary.

We see once again the affine space structure of $\mathfrak{G}_n$ over $\mathrm{H}^1(\bar{\Gamma}, \mathbb{Z}/n\mathbb{Z})$ since the difference of such cochains are morphisms.

It follows from (8) that we have:

$$\nu(\bar\gamma^{-1}) = -\nu(\bar\gamma) \tag{11}$$

Moreover, for any γ_0 in $\widetilde\Gamma$, we have $\sigma_0(\bar\gamma_0) = \gamma_0 h^k$ for some k, where as usual $\bar\gamma_0 = p^0_*(\gamma_0)$. Therefore, we have the equality $\gamma_0\sigma_0(\bar\gamma)\gamma_0^{-1} = \sigma_0(\bar\gamma_0)\sigma_0(\bar\gamma)(\sigma_0(\bar\gamma_0))^{-1}$. So by equation (9) we have

$$\sigma_0(\bar\gamma_0\bar\gamma\bar\gamma^{-1}) \;=\; \gamma_0\sigma_0(\bar\gamma)\gamma_0^{-1} \;=\; \sigma(\bar\gamma_0)\sigma_0(\bar\gamma)(\sigma(\bar\gamma_0))^{-1}$$

and this implies that

$$\nu(\bar\gamma_0\bar\gamma\bar\gamma_0^{-1}) = \nu(\bar\gamma) \tag{12}$$

The next step is now to describe the affine action of $\mathrm{Mod}^{\pm}(\Sigma) \approx \mathrm{Out}(\bar\Gamma)$ on $\mathfrak{G}_n$ in the ν-coordinates.We will do it by describing the action of any automorphism of $\bar\Gamma$.

Let $\bar f$ be a diffeomorphism of Σ, and we denote by $[\bar f]$ its isotopy class, i.e. the element of $\mathrm{Mod}^{\pm}(\Sigma) \approx \mathrm{Out}(\bar\Gamma)$ it represents. Let $[\bar f]_*$ denote one automorphism of $\bar\Gamma$ representing the action of $[\bar f]$ on $\bar\Gamma$.

Lemma 4.2. *The action of the element $[\bar f]$ of $\mathrm{Mod}^{\pm}(\Sigma)$ on $\mathfrak{G}_n$, in terms of the maps $\nu : \bar\Gamma \to \mathbb{Z}/n\mathbb{Z}$ above, is:*

$$[\bar f] * \nu = \pm\nu \circ [\bar f]_*^{-1}$$

The $\pm$ depends on whether the element $[\bar f]$ is orientation preserving or not, i.e. in $\mathrm{Mod}(\Sigma)$ (the sign is $+$), or in $\mathrm{Mod}^-(\Sigma)$ (the sign is $-$).

Observe that, due to equation (12) the term $\nu \circ [\bar f]_*^{-1}$ is well-defined despite of the fact that $[\bar f]_*$ is well-defined only up to inner automorphisms.

Proof. According to Remark 3.8 the diagonal action of the selected representative of $[\bar f]_*$ on the set of triples X defines an homeomorphism f of $M_1(\Sigma)$ that maps the geodesic flow Φ_0 onto itself. It lifts to a map $\tilde f : \widetilde M \to \widetilde M$ permuting the orbits of $\widetilde\Phi_0$. There is an automorphism $[f]_* : \widetilde\Gamma \to \widetilde\Gamma$ so that $\tilde f$ is $[f]_*$-equivariant. Given τ in $\tilde\Gamma$ then $[f]_*(\tau)$ is the only element of $\tilde\Gamma$ so that for any $x \in \widetilde M$, then

$$\tilde f(\tau(x)) \;\;=\;\; [f]_*(\tau)(\tilde f(x)).$$

Once again, $[f]_*$ is well-defined only up to inner automorphisms; we select one representative. If necessary, we change our previous choice of $[\bar f]_*$ so that it coincides with the automorphism of $\bar\Gamma$ induced by $[f]_*$.

For any element $\bar\gamma$ of $\bar\Gamma$, by definition, $\sigma_0(\bar\gamma)$ is the only element of $\widetilde\Gamma$ above $\bar\gamma$ preserving some orbit $\tilde\theta$ of $\widetilde\Phi_0$. Then, $[f]_*(\sigma_0(\bar\gamma))$ is an element of $\widetilde\Gamma$ above $[\bar f]_*(\bar\gamma)$ preserving some orbit of $\widetilde\Phi_0$, namely $\tilde f(\tilde\theta)$. This follows from the definition of $[f]_*$. Therefore:

$$\sigma_0([\bar f]_*(\bar\gamma)) = [f]_*(\sigma_0(\bar\gamma)) \tag{13}$$

Let us now consider an element of $\mathfrak{G}_n$. We have seen two ways to describe it:
– either by some morphism $\sigma : \bar\Gamma \to \hat\Gamma_n$,
– or a map $\nu : \bar\Gamma \to \mathbb{Z}/n\mathbb{Z}$.
The link between the two is given by the formula defining ν:

$$\forall\bar\gamma \in \bar\Gamma \;\; \sigma(\bar\gamma) = \bar\sigma_0(\bar\gamma)\bar h^{\nu(\bar\gamma)}$$

The morphism map $[f]_* : \tilde\Gamma \to \tilde\Gamma$ preserves the normal subgroup generated by h^n, therefore induces a morphism $[f]_{*n} : \hat\Gamma_n \to \hat\Gamma_n$.

For any element Γ of $\mathfrak{G}_n$, the image under $[\bar f]$ of Γ is the subgroup $[f]_*(\Gamma)$. It follows that if Γ corresponds to the morphism σ, then $[f]_*(\Gamma)$ corresponds to the morphism $\sigma' = [f]_{*n} \circ \sigma \circ [\bar f]_*^{-1}$: First let $\ell : \tilde\Gamma \to \hat\Gamma_n$ be the projection. We have that $\sigma(\bar\Gamma) = \ell(\Gamma)$. In addition $[f]_*(\Gamma) = \Gamma'$. So it would be natural to consider the map $[f]_{*n} \circ \sigma$ to be the morphism associated to Γ'. However if we compose $[f]_{*n} \circ \sigma$ with the projection from $\hat\Gamma_n$ to $\bar\Gamma$ we do not get the identity, we get $[\bar f]_*$. Therefore precomposition with $[\bar f]_*^{-1}$ has this property and is the map associated with Γ'.

On the other hand, let denote by ν' the map from $\bar\Gamma$ to $\mathbb{Z}/n\mathbb{Z}$ corresponding to $[f]_*(\Gamma)$. For any element $\bar\gamma$ of $\bar\Gamma$:

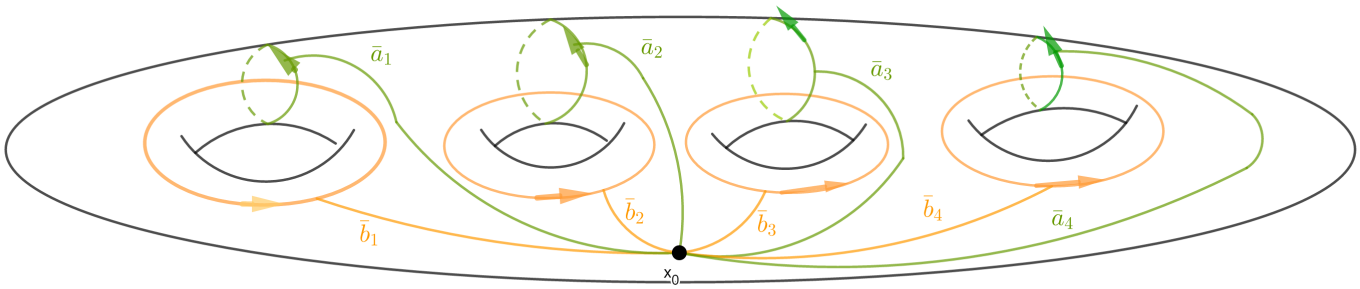


Figure 1: *A collection of oriented loops $(\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g, \bar{b}_g)$ forming a symplectic basis in homology, here for $g = 4$. The surface is considered has the boundary of an handlebody in $\mathbb{R}^3$ and the orientation of the surface is given by the normal vector pointing out the handlebody.*

$$\begin{aligned}
\bar{\sigma}_0(\bar{\gamma})\bar{h}^{\nu'(\bar{\gamma})} &= \sigma'(\bar{\gamma}) \\
&= [f]_{*n}(\sigma([\bar{f}]_*^{-1}(\bar{\gamma}))) \\
&= [f]_{*n}(\bar{\sigma}_0([\bar{f}]_*^{-1}(\bar{\gamma}))\bar{h}^{\nu([\bar{f}]_*^{-1}(\bar{\gamma}))}) \\
&= [f]_{*n}(\bar{\sigma}_0([\bar{f}]_*^{-1}(\bar{\gamma})))\bar{h}^{\pm\nu([\bar{f}]_*^{-1}(\gamma))} \text{ since } [f]_{*n} \text{ is a morphism preserving } <\bar{h}> \\
&= \bar{\sigma}_0(\bar{\gamma})\bar{h}^{\pm\nu([\bar{f}]_*^{-1}(\bar{\gamma}))} \text{ according to (13)}
\end{aligned}$$

If $\bar{f}$ preserves orientation in Σ then f_* preserves $\bar{h}$, if $\bar{h}$ reverses orientation then f_* sends $\bar{h}$ to its inverse. Therefore, the sign in $\pm\nu([\bar{f}]_*^{-1}(\bar{\gamma}))$ is given by the fact that $\bar{f}$ preserves orientation or not. The Lemma is proved. □

Let us now fix a generator system of $\bar{\Gamma}$ satisfying the usual presentation:

$$\bar{\Gamma} = \langle \bar{a}_i, \bar{b}_i \ (i = 1, ..., g) \mid [\bar{a}_1, \bar{b}_1][\bar{a}_2, \bar{b}_2]...[\bar{a}_g, \bar{b}_g] = 1 \rangle$$

More precisely, we select a base point x_0, and take such a generator system so that every $\bar{a}_i$ and $\bar{b}_i$ is represented by a loop, such that the homology classes $[\bar{a}_i]$ and $[\bar{b}_j]$ form a symplectic basis for the intersection form (i.e. the intersection numbers $[\bar{a}_i].[\bar{a}_j]$ and $[\bar{b}_i].[\bar{b}_j]$ all vanish, and $[\bar{a}_i].[\bar{b}_j]$ as well except in the case $i = j$ where we have $[\bar{a}_i].[\bar{b}_i] = +1$). We furthermore require that the only intersection between any two such loops is the base point x_0, and another single intersection point between the i-th loops $\bar{a}_i$ and $\bar{b}_i$. See Figure 1, for a collection of loops isotopic to one satisfying these properties. Denote by a_i and b_i the images of $\bar{a}_i$ and $\bar{b}_i$ by σ_0, respectively.

Lemma 4.2 describes the way that $\mathrm{Mod}^{\pm}(\Sigma)$ acts on $\mathfrak{G}_n$, and we will now do the actual computation of this action. For this, we use the explicit generating system for $\mathrm{Mod}(\Sigma)$ provided by the Lickorish Theorem. Let A_i, B_i and C_j the simple closed curves depicted in Figure 2 ($1 \leq i \leq g$, $1 \leq j \leq g-1$).

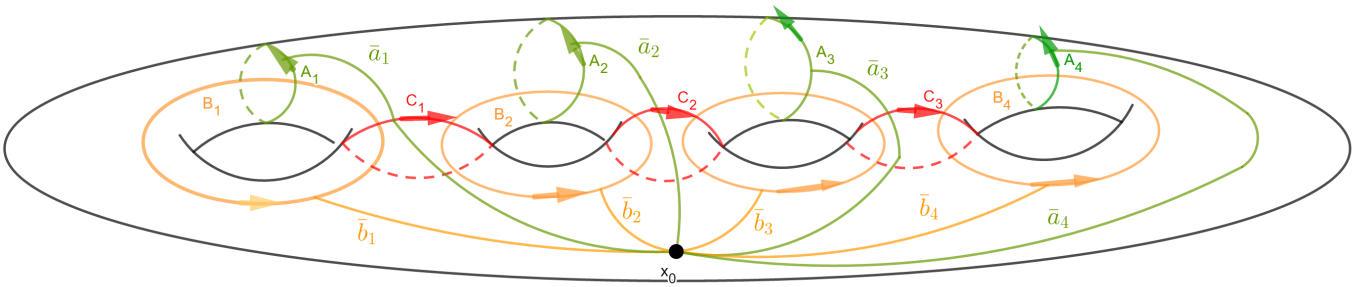


Figure 2: *According to Lickorish's Theorem, Dehn twists along these simple closed curves generate the mapping class group.*

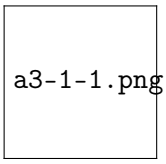


Figure 3: *Definition of* $\bar{a}_2'$, $\bar{a}_3'$, $\bar{c}_2'$.

Theorem 4.3 (Lickorish [Lic])**.** *Let* $(A_1, B_1, ..., A_g, B_g, C_1, \dots, C_{g-1})$ *the system of simple closed curves depicted in figure 2. Then, the Dehn twists along these curves generate* $\mathrm{Mod}(\Sigma)$.

Slightly abusively, we will also denote by A_i, B_i and C_j the Dehn twists along these curves, and by $[A_i]_*$, $[B_i]_*$, $[C_j]_*$ the induced automorphisms of $\bar{\Gamma} \approx \pi_1(\Sigma, x_0)$.

It follows from Figure 2 that for every i between 1 and g that $[A_i]_*$ maps every generator $\bar{a}_j$ and $\bar{b}_j$ onto itself, except $\bar{b}_i$ for which we have:

$$[A_i]_*(\bar{b}_i) = \bar{a}_i\bar{b}_i.$$

Therefore:

$$[A_i]_*^{-1}(\bar{b}_i) = \bar{a}_i^{-1}\bar{b}_i. \tag{14}$$

Similarly $[B_i]_*$ maps every generator onto itself except $\bar{a}_i$ for which we have:

$$[B_i]_*(\bar{a}_i) = \bar{a}_i\bar{b}_i^{-1}.$$

Hence:

$$[B_i]_*^{-1}(\bar{a}_i) = \bar{a}_i\bar{b}_i. \tag{15}$$

The computation of the action of $[C_i]_*$ $(1 \leq i \leq g-1)$ is more intricate. Let $\bar{c}_i$ be the element of $\bar{\Gamma}$ defined by the loop describing first the initial part of $\bar{a}_i$ until it reaches C_i, then following C_i in the direction depicted in Figure 2, then going back to x_0 by the path it uses to reach C_i. Then, $[C_i]_*$ acts trivially on every $\bar{a}_j$ for $j \neq i$, and also on every $\bar{b}_j$, except for $j = i$ and $j = i+1$. Furthermore:

$$[C_i]_*(\bar{a}_i) = \bar{c}_i\bar{a}_i\bar{c}_i^{-1}$$

$$[C_i]_*(\bar{b}_i) = \bar{c}_i\bar{b}_i$$
$$[C_i]_*(\bar{b}_{i+1}) = \bar{b}_{i+1}\bar{c}_i^{-1}$$

Therefore, the inverse of $[C_i]_*$ satisfies:

$$[C_i]_*^{-1}(\bar{a}_i) = \bar{c}_i^{-1}\bar{a}_i\bar{c}_i,\ \ [C_i]_*^{-1}(\bar{b}_i) = \bar{c}_i^{-1}\bar{b}_i,\ \ [C_i]_*^{-1}(\bar{b}_{i+1}) = \bar{b}_{i+1}\bar{c}_i. \tag{16}$$

Now we claim the following equality:

$$\bar{c}_i = \bar{a}_i\bar{b}_{i+1}^{-1}\bar{a}_{i+1}^{-1}\bar{b}_{i+1} \tag{17}$$

Indeed: introduce a new base point x_1 as depicted in Figure 3, and paths a_i', a_{i+1}' and c_i'. Let $\bar{a}_i'$ be the loop based at x_1 following first a_i', then turning around A_i, and going back to x_1 along a_i'. Define similarly $\bar{a}_{i+1}'$ and $\bar{c}_i'$. For simplicity we omit the dependence of x_1 on i.

It follows from this picture that we have $\bar{c}_i' = \bar{a}_i'(\bar{a}_{i+1}')^{-1}$.

Consider now the path ℓ going from x_0 to x_1 going along $\bar{a}_i$ until reaching C_i, and then reaching x_1 along c_i'. We isotope c_i' so that it intersects C_i at the same point that $\bar{a}_i$ does. Let $\bar{a}_i''$, $\bar{a}_{i+1}''$ and $\bar{c}_i''$ be the loops starting from x_0, following ℓ, turning along $\bar{a}_i'$, $\bar{a}_{i+1}'$ and $\bar{c}_i'$ respectively before going back to x_0 along ℓ. Clearly, $\bar{a}_i''$ is homotopic to $\bar{a}_i$, and $\bar{c}_i''$ is homotopic to $\bar{c}_i$. Therefore, up to homotopy (based at x_0):

$$\bar{c}_i \approx \bar{a}_i(\bar{a}_{i+1}'')^{-1}$$

But the same figure shows that $\bar{a}_{i+1}$ is homotopic to $\bar{b}_{i+1}\bar{a}_{i+1}''\bar{b}_{i+1}^{-1}$.

The equality (17) follows.

We can now compute how the generators A_i, B_i, C_i acts on $\mathfrak{G}_n$.

Proposition 4.4. *Let i be an integer between 1 and g. The action of the element $[A_i]$ of $\mathrm{Mod}(\Sigma)$ represented by the Dehn twist A_i on $\mathfrak{G}_n$ is given by:*

$$[A_i] * \nu(\bar{a}_j) = \nu(\bar{a}_j)\ \ \forall j, \quad [A_i] * \nu(\bar{b}_j) = \nu(\bar{b}_j)\ \ \forall j \neq i,\ [A_i] * \nu(\bar{b}_i) = -\nu(\bar{a}_i) + \nu(\bar{b}_i).$$

The action of the element $[B_i]$ of $\mathrm{Mod}(\Sigma)$ represented by B_i is:

$$[B_i] * \nu(\bar{b}_j) = \nu(\bar{b}_j)\ \ \forall j, \quad [B_i] * \nu(\bar{a}_j) = \nu(\bar{a}_j)\ \ \forall j \neq i,\ [B_i] * \nu(\bar{a}_i) = \nu(\bar{a}_i) + \nu(\bar{b}_i).$$

Finally, for i between 1 and $g-1$, the action of the element $[C_i]$ of $\mathrm{Mod}(\Sigma)$ represented by C_i is:

$$[C_i] * \nu(\bar{a}_j) = \nu(\bar{a}_j)\ \ \forall j, \quad [C_i] * \nu(\bar{b}_j) = \nu(\bar{b}_j)\ \ \forall j \neq i, i+1,$$

and

$$[C_i] * \nu(\bar{b}_i) = \nu(\bar{b}_i) - \nu(\bar{a}_i) + \nu(\bar{a}_{i+1}) + 1,\ \ [C_i] * \nu(\bar{b}_{i+1}) = \nu(\bar{b}_{i+1}) + \nu(\bar{a}_i) - \nu(\bar{a}_{i+1}) - 1$$

Proof. Recall that according to Lemma 4.2, for every element $[\bar{f}]$ of $\mathrm{Mod}(\Sigma)$ we have:

$$[\bar{f}] * \nu = \nu \circ [\bar{f}]_*^{-1}.$$

Most formulae in the statement of the Proposition immediatly follow. The only cases we have to consider are:

$$\begin{aligned}
[A_i] * \nu(\bar{b}_i) &= \nu(\bar{a}_i^{-1}\bar{b}_i)\ \text{(because of Equation (14))} \\
&= -\nu(\bar{a}_i) + \nu(\bar{b}_i) - c(\bar{a}_i^{-1}, \bar{b}_i)\ \ \text{(because of Equation (10))} \\
&= -\nu(\bar{a}_i) + \nu(\bar{b}_i)\ \ \text{(because of Lemma 3.2)}
\end{aligned}$$

The formula for $[B_i] * \nu(\bar{a}_i)$ is similar.

The remaining computations we have to do are for $[C_i]_*\beta(\bar{a}_i)$, $[C_i]_*\beta(\bar{b}_i)$, and $[C_i]_*\beta(\bar{b}_{i+1})$. Concerning the first, we have:

$$[C_i] * \nu(\bar{a}_i) = \nu(\bar{c}_i^{-1}\bar{a}_i\bar{c}_i) = \nu(\bar{a}_i)$$

Thanks to Equations (16), (10) one gets:

$$[C_i] * \nu(\bar{b}_i) = \nu(\bar{b}_i) - \nu(\bar{c}_i) - c(\bar{c}_i^{-1}, \bar{b}_i),\ \ [C_i] * \nu(\bar{b}_{i+1}) = \nu(\bar{b}_{i+1}) + \nu(\bar{c}_i) - c(\bar{b}_{i+1}, \bar{c}_i).$$

But it follows from Lemma 3.2 that $c(\bar{c}_i^{-1}, \bar{b}_i)$ and $c(\bar{b}_{i+1}, \bar{c}_i)$ vanishes. Therefore:

$$[C_i] * \nu(\bar{b}_i) = \nu(\bar{b}_i) - \nu(\bar{c}_i),\ \ [C_i] * \nu(\bar{b}_{i+1}) = \nu(\bar{b}_{i+1}) + \nu(\bar{c}_i) \tag{18}$$

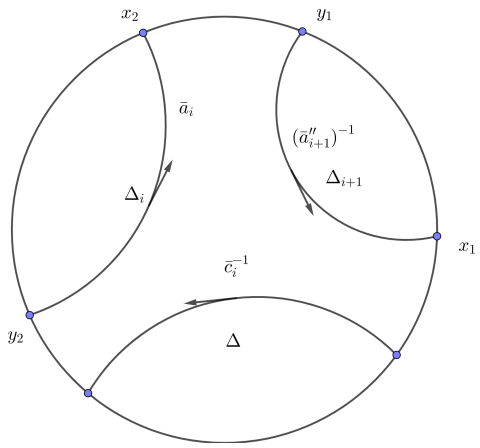


Figure 4: Δ, Δ_i *and* Δ_{i+1} *are preserved by respectively* $\bar{c}^{-1}$, $\bar{a}_i$ *and* $(\bar{a}''_{i+1})^{-1}$. *The attracting fixed point of* $(\bar{a}''_{i+1})^{-1}$ *is* x_1, *and the attracting fixed point of* $\bar{a}_i$ *is* x_2.

Hence, the key point is to compute $\nu(\bar{c}_i)$. Since $\bar{c}_i = \bar{a}_i(\bar{a}''_{i+1})^{-1}$, we have:

$$\nu(\bar{c}_i) = \nu(\bar{a}_i) - \nu(\bar{a}''_{i+1}) - c(\bar{a}_i, (\bar{a}''_{i+1})^{-1})$$

Hence, we have to compute $c(\bar{a}_i, (\bar{a}''_{i+1})^{-1})$. Observe that the closed simple curves A_i, C_i and A_{i+1} form the boundary of a pair of pants. If we realize it as closed geodesics for the hyperbolic metric, we see that they are the projections of the axis of respectively $\bar{a}_i$, $\bar{c}_i$ and $(\bar{a}''_{i+1})^{-1}$ (recall Figure 3). It follows that the axis of these elements in $\mathbb{H}^2$ have the configuration illustrated in Figure 4:

Consider now the lifted action in $\widetilde{\mathbb{R}P^1}$ is as in Figure 5. Here a_i and a''_{i+1} are the lifts of $\bar{a}_i$ and $\bar{a}''_{i+1}$ with fixed points in $\widetilde{\mathbb{R}P^1}$. Here x_2, y_2 are fixed by $\bar{a}_i$ with x_2 the attracting fixed point. This generates a_i fixing $\widetilde{x}_2, \widetilde{y}_2$ (and infinitely many other pairs). Similarly $(\bar{a}''_{i+1})^{-1}$ fixes x_1, y_1 and so on.

Claim: for every $\tilde{x}$ in $\widetilde{\mathbb{R}P^1}$ we have $a_i(a''_{i+1})^{-1}(\tilde{x}) > \tilde{x}$.

Let us prove this claim. It is enough to prove it for every $\tilde{x}$ in the interval $[\tilde{y}_1, h(\tilde{y}_1)]$.

- **If $\tilde{x}$ lies in $[\tilde{y}_1, \tilde{x}_2]$:** Then, we have $(a''_{i+1})^{-1}(\tilde{x}) > \tilde{x}$. If $(a''_{i+1})^{-1}(\tilde{x})$ lies in $[\tilde{y}_1, \tilde{x}_2]$, then its image by a_i increases it, so the final image is bigger than $\tilde{x}$. If not, then $(a''_{i+1})^{-1}(\tilde{x}) \geq \tilde{x}_2$, and therefore $a_i(a''_{i+1})^{-1}(\tilde{x})$ remains bigger or equal than $\tilde{x}$. The Claim is true in this case.
- **If $\tilde{x}$ lies in $[\tilde{x}_2, \tilde{y}_2]$:** The interval $[\tilde{x}_2, \tilde{y}_2]$ is contained in the attracting region of $h(\tilde{x}_1)$ under $(a''_{i+1})^{-1}$ we therefore have $(a''_{i+1})^{-1}(\tilde{x}_2) > \tilde{x}_2$ and $(a''_{i+1})^{-1}(\tilde{y}_2) > \tilde{y}_2$. But since Δ_i is the lifting of a simple closed geodesic, $\tilde{y}_2$ cannot be between $a_i(\tilde{x}_2)$ and $a_i(\tilde{y}_2)$. Therefore, we have $(a''_{i+1})^{-1}(\tilde{x}_2) > \tilde{y}_2$: the interval $(a''_{i+1})^{-1}([\tilde{x}_2, \tilde{y}_2])$ is contained in the region $[\tilde{y}_2, h(\tilde{x}_1))]$ where we have $a_i(\tilde{x}) > \tilde{x}$. Therefore, the Claim is still true there.
- **If $\tilde{x}$ lies in $[\tilde{y}_2, h(\tilde{x}_1)]$:** then $(a''_{i+1})^{-1}(\tilde{x})$ lies in $[\tilde{x}, h(\tilde{x}_1)]$, and a_i is increasing there, so the Claim is true in this case too.
- **If $\tilde{x}$ lies in $[h(\tilde{x}_1), h(\tilde{y}_1)]$:** Then, $(a''_{i+1})^{-1}(\tilde{x})$ remains in this interval. Therefore, its image under a_i is bigger than $\tilde{x}$. Notice that a_i is increasing in the interval $[h(\tilde{x}_1, h(\tilde{y}_1)]$. As in the second case we cannot have crossing of geodesics, so in fact for every $\tilde{x}$ in $[h(\tilde{x}_1, h(\tilde{y}_1]$ then $a_i(a''_{i+1})^{-1}(\tilde{x}) > h(\tilde{y}_1)$. The Claim follows once more.

The Claim is proved.

According to the Claim, the cocycle $c(\bar{a}_i, (\bar{a}''_{i+1})^{-1})$ is positive. In addition $a_i(\tilde{x}_1) < h(\tilde{x}_1)$, hence $c(\bar{a}_i, (\bar{a}''_{i+1})^{-1}) \leq 1$. It follows $c(\bar{a}_i, (\bar{a}''_{i+1})^{-1}) = 1$.

Therefore:

$$\nu(\bar{c}_i) = \nu(\bar{a}_i) - \nu(\bar{a}''_{i+1}) - 1$$

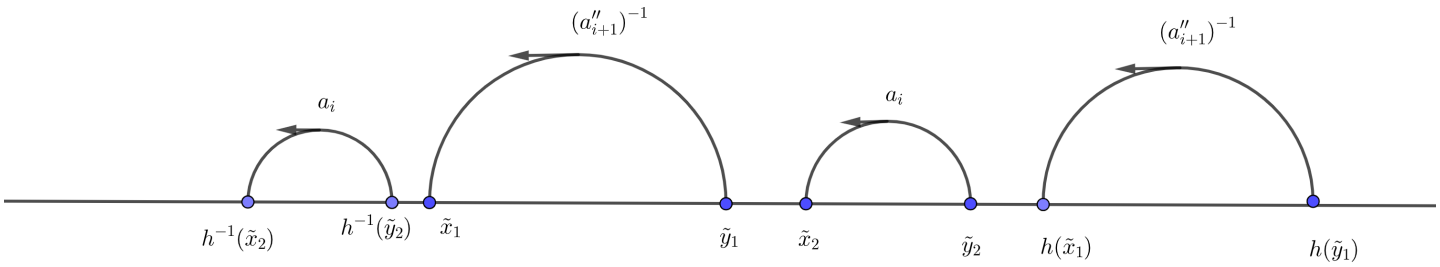


Figure 5: *Liftings in* $\widetilde{\mathbb{R}P^1}$. *We have depicted two attracting fixed points* $\tilde{x}_1$ *and* $h(\tilde{x}_1)$ *of* $(a''_{i+1})^{-1}$, *and two attracting fixed points* $\tilde{x}_2$ *and* $h^{-1}(\tilde{x}_2)$ *of* a_i.

and since $\bar{a}''_{i+1}$ is conjugated to $\bar{a}_{i+1}$, we get:

$$\nu(\bar{c}_i) = \nu(\bar{a}_i) - \nu(\bar{a}_{i+1}) - 1$$

Equations (18) become:

$$[C_i] * \nu(\bar{b}_i) = \nu(\bar{b}_i) - \nu(\bar{a}_i) + \nu(\bar{a}_{i+1}) + 1, \;\; [C_i] * \nu(\bar{b}_{i+1}) = \nu(\bar{b}_{i+1}) + \nu(\bar{a}_i) - \nu(\bar{a}_{i+1}) - 1$$

Proposition 4.4 is proved. □

Equation (10) implies that the map $\nu : \bar{\Gamma} \to \mathbb{Z}/n\mathbb{Z}$ describing an element of $\mathfrak{G}_n$ is characterized by the values it takes on the generating set $\bar{a}_i, \bar{b}_i$ $(i = 1, ..., g)$. In other words, one can parametrize $\mathfrak{G}_n$ by the n-tuples $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ in $(\mathbb{Z}/n\mathbb{Z})^{2g}$ where α_i is the value taken by ν at $\bar{a}_i$, and β_i the value it takes at $\bar{b}_i$. Observe that all morphisms are possible since one can add to any ν any morphism from $\bar{\Gamma}$ into $\mathbb{Z}/n\mathbb{Z}$. In particular, all the α_i and β_i can vanish, this corresponds to the base group Γ.

Let us decompose $\mathfrak{G}_n$ as a sum $V_1 \oplus \ldots \oplus V_g$ where every V_i is made of elements with coordinates $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ (where $\alpha_i, \beta_i \in \mathbb{Z}/n\mathbb{Z}$) satisfying $\alpha_j = \beta_j = 0$ for all $j \neq i$. According to Proposition 4.4, this decomposition is invariant by the subgroup G of $\mathrm{Mod}(\Sigma)$ generated by the $[A_i]$'s and the $[B_i]$'s. More precisely, in every V_i, the actions of $[A_i]$ and $[B_i]$ are given by the matrices:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

In particular, the action of G preserves $(0, ..., 0)$: it is linear. Moreover, these two matrices generate the entire $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$, therefore, any element of $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})^g$ is realized by an element of G.

On the other hand, Proposition 4.4 shows that for every i between 1 and $g-1$, the action of $[C_i]$ is trivial on V_j for $j \neq i, i+1$, and that on $V_i \oplus V_{i+1}$ this action is given by:

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & |0 \\ -1 & 1 & 1 & 0 & |1 \\ 0 & 0 & 1 & 0 & |0 \\ 1 & 0 & -1 & 1 & |-1 \end{array}\right)$$

This action is not linear, and its translation part $\tau([C_i]_*)$ in the coordinate system $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ is the vector satisfying $\beta_i = -\beta_{i+1} = 1$ for all i, with all other coordinates vanishing.

We can now prove our Main Theorem:

Theorem 4.5. *Let $q_0 : M \to M_1(\Sigma)$ be a finite covering of degree n along the fibers. Then, if n is odd, there is only one Anosov flow on M up to orbital equivalence. If n is even, there are exactly two orbital equivalence classes of Anosov flows.*

Remark 4.6. We thank J. Bowden who had indicated to us that this Theorem is closely related to Theorem 2.9 in [Rand], but we point out that our result concerns the case where Σ is a closed surface, whereas Randal-Williams only considers the case of compact surfaces with non-empty boundary.

Proof. The elements of $\mathfrak{G}_n$ are parametrized by $2g$ tuples $(\alpha_1, \beta_1, ..., \alpha_g, \beta_g)$, with entries in $\mathbb{Z}/n\mathbb{Z}$. We first show that any $\mathrm{Mod}(\Sigma)$ orbit contains an element of the form $(0, 0, ..., 0, \beta)$ for some β in $\mathbb{Z}/n\mathbb{Z}$. From this, we show its orbit also has the element $(0, 0, ..., 0, \beta + 2)$. This reduces the number of orbits to 2. Then we analyze separately the cases of n odd and n even, with n even being trickier.

Since any element of $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})^g$ is realized by an element of G, every element of $\mathfrak{G}_n$ admits in its G-orbit an element of the form:

$$(0, \beta_1, 0, \beta_2, \ldots, 0, \beta_g) \tag{19}$$

Let us now consider the subgroup H generated by the $g-1$ elements $C_1, \ldots, C_{g-1}$. Since all the curves $\bar{c}_i$ are disjoint, it is an abelian group. The action of some element $C_1^{k_1} C_2^{k_2} \ldots C_{g-1}^{k_{g-1}}$ is:

$$\begin{array}{rcl} \alpha_i & \to & \alpha_i \\ \beta_i & \to & \beta_i + k_i\alpha_{i+1} - (k_i + k_{i-1})\alpha_i + k_{i-1}\alpha_{i-1} + k_i - k_{i-1} \end{array}$$

where we adopt the convention $k_0 = k_g = 0$. This action does not change the value of each α_i, and moreover one can check that the sum $\sum_{i=1}^{g} \beta_i$ is constant along the orbits of H. We do not do an explicit computation here, which is fairly simple. Anyway, we see that by applying a well-suited element of H, we can transform the element of the form (19) to another one where all the α_i's and β_i's vanish, except β_g, since every α_i remains null and every β_i is changed by adding $k_i - k_{i-1}$. In other words, every element of $\mathfrak{G}_n$ admits in its $\mathrm{Mod}(\Sigma)$-orbit an element of the form:

$$(0, 0, \ldots, 0, 0, 0, \beta) \tag{20}$$

Let us go back, applying C_{g-1}, we get:

$$(0, 0, \ldots, 0, 1, 0, \beta - 1)$$

Now use an element of G mapping this element to:

$$(0, 0, \ldots, 0, 1, \beta - 1, 0)$$

Apply C_{g-1}: we now get:

$$(0, 0, \ldots, 0, \beta + 1, \beta - 1, -\beta)$$

Since $\beta - 1$ and $-\beta$ are relatively prime, applying an element of G, composition of A_g's and B_g's, we obtain:

$$(0, 0, \ldots, 0, \beta + 1, 0, 1)$$

and by applying $C_{g-1}^{-1-\beta}$, we get:

$$(0, 0, \ldots, 0, 0, 0, \beta + 2)$$

In summary, any element of $\mathfrak{G}_n$ admits in its $\mathrm{Mod}(\Sigma)$-orbit an element of the form $(0,0,\dots,0,0,0,\beta)$, and admits also all the elements of the form $(0,0,\dots,0,0,0,\beta+2k)$ where k is any integer.

It follows that when n is odd, in which case 2 is a generator of $\mathbb{Z}/n\mathbb{Z}$, that every $\mathrm{Mod}(\Sigma)$-orbit in $\mathfrak{G}_n$ contains $(0,0,\dots,0,0,0,0)$. In other words, there is only one $\mathrm{Mod}(\Sigma)$-orbit; hence only one $\mathrm{Mod}^{\pm}(\Sigma)$-orbit. The theorem follows in this case from Proposition 3.17.

Let us now assume that n is even. The same argument as above shows that there are at most two $\mathrm{Mod}(\Sigma)$-orbits in $\mathfrak{G}_n$: the orbit of $(0,0,\dots,0,0,0,0)$ and the orbit of $(0,0,\dots,0,0,0,1)$. The proof of the Theorem will be finished if we prove that these two orbits are disjoint.

Since n is even, every α_i and β_i, which is an integer modulo n, defines an element $\bar{\alpha}_i$ and $\bar{\beta}_i$ of $\mathbb{Z}/2\mathbb{Z}$. In other words, there is a well-defined morphism $\mathfrak{G}_n \to \mathfrak{G}_2$. This morphism is $\mathrm{Mod}^{\pm}(\Sigma)$-equivariant – this follows from Proposition 4.4 which shows that the action in $\mathfrak{G}_2$ is obtained from taking the action on $\mathfrak{G}_n$ and projecting it to modulo 2. Therefore, if the $\mathrm{Mod}^{\pm}(\Sigma)$-orbits of $(0,0,\dots,0,0,0,0)$ and of $(0,0,\dots,0,0,0,1)$ are different in $\mathfrak{G}_2$, they are also different in $\mathfrak{G}_n$.

Hence, in order to achieve the proof of the Theorem, we just have to consider the case $n=2$.

For every element of $\mathfrak{G}_2 = V_1 \oplus \dots \oplus V_g$, let us call *vanishing number* the number of indices i for which the components in V_i are zero. More precisely, the vanishing number is this integer modulo 2. For example, the vanishing number for $(0,0,\dots,0,0,0,0)$ is the class modulo 2 of 0, whereas the vanishing number for $(0,0,\dots,0,0,0,1)$ is the class modulo 2 of 1: they are different.

Claim: The vanishing number is constant along $\mathrm{Mod}(\Sigma)$-orbits.

Let us prove the claim: clearly, the vanishing number does not change under the action of A_i or B_i, since they act in V_i as elements of $\mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$.

Let us consider C_i $(1 \le i \le g-1)$. It acts trivially on every V_j except maybe V_i and V_{i+1}. We refer to the explicit formula for the action of C_i on $(\alpha_i,\beta_i,\alpha_{i+1},\beta_{i+1})$ given in the beginning of the proof of this theorem.

– If $\alpha_i = \alpha_{i+1} = 1$, the components in V_i and V_{i+1} are nonzero, and the same is true after applying C_i since we still have after the action $\alpha_i = \alpha_{i+1} = 1$: the vanishing number remains the same.

– If $\alpha_i \neq \alpha_{i+1}$, then, since we are in $\mathbb{Z}/2\mathbb{Z}$, we have $\alpha_{i+1} - \alpha_i + 1 = 0$, and it follows that C_i acts trivially on such an element: the vanishing number remains the same in this case too.

– If $\alpha_i = \alpha_{i+1} = 0$: then we have:

$$C_i(0,\beta_i,0,\beta_{i+1}) = (0,\beta_i+1,0,\beta_{i+1}+1)$$

Hence, if $\beta_i = \beta_{i+1} = 0$, or if $\beta_i = \beta_{i+1} = 1$, the vanishing number before and after C_i differ by 2, hence is the same modulo 2. If $\beta_i \neq \beta_{i+1}$, one component vanishes and not the other, and the same is true after applying C_i.

We have proved the Claim. Since the vanishing numbers of $(0,0,\dots,0,0,0,0)$ and $(0,0,\dots,0,0,0,1)$ are different, they cannot be in the same $\mathrm{Mod}(\Sigma)$-orbit.

The only remaining step is to show that the vanishing number is also preserved by orientation reversing elements of $\mathrm{Mod}^{\pm}(\Sigma)$. For this, we just have to prove that it is true for *one* of them. Let us consider the horizontal plane P such that the surface is symetric relatively to the symetry s in P. We can isotop every B_i such that they are all contained in P, hence preserved by s (including their orientation), and such that every A_i is orthogonal to P and preserved by s, but with the reversed orientation. We refer to figure 2: we can choose s so that it preserves the loops A_i, B_i in figure 2. One can assume, and we do, that the base point x_0 is in P, hence fixed by s, and that for every i, the portion $\hat{a}_i$ (respectively $\hat{b}_i$) of the loop $\bar{a}_i$ (respectively $\bar{b}_i$) joining x_0 to A_i (respectively B_i) lies above P, and, in addition, in the case of $\bar{a}_i$, this portion reaches A_i at its intersection with P.

Then, the composition of $\hat{a}_i$ (respectively $\hat{b}_i$) with its image under s provides a loop $\hat{a}_i^s$ (respectively $\hat{b}_i^s$) based at x_0. It follows from these choices that $\bar{b}_i$ (respectively $\bar{a}_i$) is conjugated by $\hat{b}_i^s$ (respectively $\hat{a}_i^s$) to itself (respctively to its inverse).

Therefore, according to Lemma 4.2, and since s is orientation reversing, the action of $[s]$ on $\mathfrak{G}_2$ (or $\mathfrak{G}_n$) is given by, for all j:



Figure 6: $g-1$ *non separating curves on the surface.*

$$[s] * \nu(a_j) = \nu(a_j)$$
$$[s] * \nu(\bar{b}_j) = -\nu(\bar{b}_j)$$

This obviously preserves the vanishing number. Thus concludes the proof of Theorem 4.5. □

We end this Section with a proposition showing that the image in $\mathrm{Aff}(\mathfrak{G}_n)$ of $\mathrm{Mod}(\Sigma)$ is isomorphic to $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$.

Proposition 4.7. *The Torelli group acts trivially on* $\mathfrak{G}_n$.

Proof. By definition, an element of the Torelli group is an element of $\mathrm{Mod}^{\pm}(\Sigma)$ which acts trivially in homology. In particular it is orientation preserving. It also follows that its linear part is the identity. But it could still act on $\mathfrak{G}_n$ as a translation. However, by a Theorem of Johnson ([Joh]) combined with a Theorem of Powell in the case $g = 2$ ([Pow]), the Torelli group is generated by:

– Dehn twists along separating non-homotopically trivial simple closed curves in the case $g = 2$,

– compositions $T_1 T_2^{-1}$ where T_1 and T_2 are left Dehn twists along simple closed curves that are boundary of a compact surface of genus 1 embedded in Σ (in the case $g \geq 3$).

Consider the red simple closed curves $D_1, \ldots, D_{g-1}$ depicted in Figure 6. In the case $g = 2$, there is only one such a curve D_1, and every separating, homotopically non-trivial simple closed curve in Σ is the image of D_1 under some homeomorphism. Therefore, it follows that in this case the Torelli group is generated by the conjugates of T_1.

In the case $g \geq 3$ any pair of simple closed curves bounding a genus 1 surface embedded in Σ is the image under some homeomorphism of the region of Σ between two successive loops D_i and D_{i+1}. Therefore, the Torelli group is generated by conjugates of one of the $T_i T_{i+1}^{-1}$.

Therefore, in order to prove the proposition we just have to show that every T_i acts trivially on $\mathfrak{G}_n$.

But every D_i are disjoint from the loops A_i and B_i. Therefore the Dehn twists along D_i preserves the conjugacy classes of $\bar{a}_i$ and $\bar{b}_i$. It follows that every T_i acts trivially on $\mathfrak{G}_n$.

□

Remark 4.8. According to proposition 4.7, the action of $\mathrm{Mod}(\Sigma)$ on $\mathfrak{G}_n$ induces a (faithful) affine action of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ on the torus $\mathfrak{G}_n \approx (\mathbb{Z}/n\mathbb{Z})^{2g}$ whose linear part is the canonical representation of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. But this action is affine and admits a non-trivial translation part $[\tau]$, that represents a non-trivial element of $H^1(\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z}), (\mathbb{Z}/n\mathbb{Z})^{2g})$.

This feature is in contrast with the fact that $H^1(\mathrm{Sp}(2g, \mathbb{Z}), (\mathbb{Z})^{2g})$ vanishes.

Remark 4.9. In [Gi], E. Giroux considered a related problem: the classification of contact structures on the circle bundle M up to isotopy and up to conjugation by a diffeomorphism. A case of interest for us is the case of universally tight contact structures, with wrapping number $-n$. E. Giroux shows that they are all isotopic to the pull-back $q^*\alpha_0$ by some finite covering $q : M \to M_1(\Sigma)$ along the fibers, where α_0 is the contact 1-form on $M_1(\Sigma)$ whose Reeb flow is the geodesic flow (for some hyperbolic Riemannian metric). According to Ghys' Theorem (see Theorem 3.3), Anosov flows on M are always isotopic to the Reeb flow of such a structure. Clearly, an isotopy between contact structures provides an isotopy between their Reeb flows, i.e. Anosov flows, and vice-versa. Similarly, there is a one-to-one correspondence between conjugacy classes of these contact structures and orbital equivalence classes of Anosov foliations.

As a matter of fact, Giroux obtains a classification of isotopy classes similar as we do here: the action of K_1 on these contact structures is simply transitive. Actually, it follows from Giroux's work and the results in the present paper that these two classification problems are equivalent.

Giroux furthermore claims to have classified conjugacy classes, and to show that the number of conjugacy classes of these contact structures is the number $\varphi(n)$ of divisors of n ([Gi, Theorem 3.1 and Proposition 3.10]). Hence, this statement is in contradiction with our Theorem 4.5.

The point is that Giroux's proof is not completely correct at its concluding step: in the proof of Proposition 3.10, he implicitly assumes that the action of $\mathrm{Mod}(\Sigma)$ on the space $\mathfrak{G}_n$ of index n subgroups along the fiber is linear, whereas this action is genuinely affine, as shown in the present article.

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